16. DEFINITE INTEGRALS

Fundamental Theorem of Integral Calculus

Let $f(x)$ be a continuous function defined on an interval $[a, b]$ and let the antiderivative of $f(x)$ be $F(x)$. Then, the definite integral of $f(x)$ over $[a, b]$, denoted by $\int_a^b f(x)\,dx$, is given by

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

NOTE Here, a and b are respectively known as the *lower limit* and the *upper limit* of the integral.

The value of a definite integral is unique, for, if $\int f(x)\,dx = F(x) + C$ then

$$\int_a^b f(x)\,dx = [F(x) + C]_a^b = \{F(b) + C\} - \{F(a) + C\} = F(b) - F(a).$$

SOLVED EXAMPLES

[Integrals based on formulae and integration by parts]

EXAMPLE 1 *Evaluate:*

(i) $\int_2^4 \frac{dx}{x}$ *(ii)* $\int_4^9 \sqrt{x}\,dx$ *(iii)* $\int_0^2 \sqrt{6x+4}\,dx$

(iv) $\int_0^1 \frac{dx}{\sqrt{5x+3}}$ *(v)* $\int_1^{\sqrt{2}} \frac{dx}{x(\sqrt{x^2-1})}$ *(vi)* $\int_0^{\pi} \sin 5x\,dx$

(vii) $\int_0^{\pi/2} \cos^2 x\,dx$ **[CBSE 2002]** *(viii)* $\int_0^{\pi/4} \tan^2 x\,dx$

(ix) $\int_0^{\pi/4} \sin 2x \sin 3x\,dx$ **[CBSE 2006]**

SOLUTION (i) $\int_2^4 \frac{dx}{x} = [\log x]_2^4 = (\log 4 - \log 2) = (2\log 2 - \log 2) = \log 2.$

(ii) $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = \frac{2}{3}\cdot[(9)^{3/2} - (4)^{3/2}] = \frac{38}{3}.$

(iii) $\int_0^2 \sqrt{6x+4}\,dx = \left[\frac{2}{3}\cdot\frac{(6x+4)^{3/2}}{6}\right]_0^2 = \frac{1}{9}\cdot[(16)^{3/2}-(4)^{3/2}] = \frac{56}{9}.$

(iv) $\int_0^1 \frac{1}{\sqrt{5x+3}}\,dx = \int_0^1 (5x+3)^{-1/2}dx = \left[2\cdot\frac{(5x+3)^{1/2}}{5}\right]_0^1$

$= \frac{2}{5}(\sqrt{8}-\sqrt{3}).$

(v) $\int_1^{\sqrt{2}} \frac{dx}{x(\sqrt{x^2-1})} = [\sec^{-1}x]_1^{\sqrt{2}} = [\sec^{-1}(\sqrt{2}) - \sec^{-1}(1)] = \left(\frac{\pi}{4}-0\right) = \frac{\pi}{4}.$

(vi) $\int_0^{\pi} \sin 5x\,dx = \left[\frac{-\cos 5x}{5}\right]_0^{\pi} = -\frac{1}{5}[\cos 5\pi - \cos 0] = \frac{2}{5}.$

(vii) $\int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)\,dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4}.$

(viii) $\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4}(\sec^2 x - 1)\,dx = [\tan x - x]_0^{\pi/4} = \left(1-\frac{\pi}{4}\right).$

(ix) $\int_0^{\pi/4} \sin 2x \sin 3x\,dx = \frac{1}{2}\int_0^{\pi/4}(2\sin 2x \sin 3x)\,dx$

$= \frac{1}{2}\int_0^{\pi/4}(\cos x - \cos 5x)\,dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\sin\frac{\pi}{4} - \frac{\sin(5\pi/4)}{5}\right)\right] = \frac{3\sqrt{2}}{10}.$

EXAMPLE 2 *Evaluate:* (i) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx$ (ii) $\int_0^{\pi/2} \sqrt{1+\cos 2x}\,dx$

SOLUTION (i) $\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx = \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x + 2\sin x\cos x}\,dx$

$= \int_0^{\pi/4}(\cos x + \sin x)\,dx = [\sin x - \cos x]_0^{\pi/4} = 1.$

(ii) $\int_0^{\pi/2} \sqrt{1+\cos 2x}\,dx = \int_0^{\pi/2} \sqrt{2\cos^2 x}\,dx$

$= \sqrt{2}\int_0^{\pi/2} \cos x\,dx = \sqrt{2}[\sin x]_0^{\pi/2} = \sqrt{2}.$

EXAMPLE 3 *Evaluate:* (i) $\int_0^{\pi/2} \cos^3 x\, dx$ (ii) $\int_0^{\pi/2} \sin^4 x\, dx$

SOLUTION (i) $\int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\pi/2} \left(\frac{3\cos x + \cos 3x}{4} \right) dx$

$[\because \quad \cos 3x = 4\cos^3 x - 3\cos x]$

$$= \frac{3}{4} \cdot \int_0^{\pi/2} \cos x\, dx + \frac{1}{4} \cdot \int_0^{\pi/2} \cos 3x\, dx$$

$$= \frac{3}{4} \cdot [\sin x]_0^{\pi/2} + \frac{1}{4} \cdot \left[\frac{\sin 3x}{3} \right]_0^{\pi/2}$$

$$= \left(\frac{3}{4} - \frac{1}{12} \right) = \frac{8}{12} = \frac{2}{3}.$$

(ii) $\int_0^{\pi/2} \sin^4 x\, dx = \frac{1}{4} \int_0^{\pi/2} (2\sin^2 x)^2 dx$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} (1 - \cos 2x)^2 dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} (1 - 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} \left[1 - 2\cos 2x + \frac{(1 + \cos 4x)}{2} \right] dx$$

$$= \frac{1}{4} \cdot \int_0^{\pi/2} \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \right) dx$$

$$= \frac{3}{8} \cdot \int_0^{\pi/2} dx - \frac{1}{2} \int_0^{\pi/2} \cos 2x\, dx + \frac{1}{8} \int_0^{\pi/2} \cos 4x\, dx$$

$$= \frac{3}{8} \cdot [x]_0^{\pi/2} - \frac{1}{2} \cdot \left[\frac{\sin 2x}{2} \right]_0^{\pi/2} + \frac{1}{8} \cdot \left[\frac{\sin 4x}{4} \right]_0^{\pi/2}$$

$$= \left(\frac{3\pi}{16} - 0 + 0 \right) = \frac{3\pi}{16}.$$

EXAMPLE 4 *Evaluate:* (i) $\int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}}$ (ii) $\int_0^1 \frac{dx}{(1 + x + x^2)}$

SOLUTION (i) $\int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}} = \int_0^4 \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$

$$= \left[\log \left| (x+1) + \sqrt{x^2+2x+3} \right| \right]_0^4$$

$$= \{\log |5+\sqrt{27}| - \log |1+\sqrt{3}|\}.$$

(ii) $$\int_0^1 \frac{dx}{(1+x+x^2)} = \int_0^1 \frac{dx}{\left[\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}\right]} = \int_0^1 \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}$$

$$= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{\left(x+\frac{1}{2}\right)}{(\sqrt{3}/2)} \right]_0^1 = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$= \frac{2}{\sqrt{3}} \cdot \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi}{3\sqrt{3}}.$$

EXAMPLE 5 *Evaluate:*

(i) $\int_0^a \frac{dx}{\sqrt{ax-x^2}}$ *(ii)* $\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx$

SOLUTION (i) $$\int_0^a \frac{dx}{\sqrt{ax-x^2}} = \int_0^a \frac{dx}{\sqrt{-\left(x^2-ax+\frac{a^2}{4}\right)+\frac{a^2}{4}}}$$

$$= \int_0^a \frac{dx}{\sqrt{\left(\frac{a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2}}$$

$$= \left[\sin^{-1} \frac{\left(x-\frac{a}{2}\right)}{\left(\frac{a}{2}\right)} \right]_0^a = \left[\sin^{-1}\left(\frac{2x-a}{a}\right) \right]_0^a$$

$$= [\sin^{-1}(1) - \sin^{-1}(-1)]$$

$$= 2\sin^{-1}(1) = \left(2 \times \frac{\pi}{2}\right) = \pi.$$

(ii) $$\int_0^{\sqrt{2}} \sqrt{2-x^2}\, dx = \int_0^{\sqrt{2}} \sqrt{(\sqrt{2})^2-x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{2-x^2} + \frac{(\sqrt{2})^2}{2} \cdot \sin^{-1}\frac{x}{\sqrt{2}} \right]_0^{\sqrt{2}}$$

$$= [0+\sin^{-1}(1)] - [0+\sin^{-1}0] = \frac{\pi}{2}.$$

EXAMPLE 6 *Evaluate:*

(i) $\int_0^{\pi/2} x\cos x\,dx$ *(ii)* $\int_0^{\pi} \cos 2x \log \sin x\,dx$

(iii) $\int_1^2 \frac{\log x}{x^2}\,dx$ *(iv)* $\int_0^{\pi/6} (2+3x^2)\cos 3x\,dx$

SOLUTION (i) Integrating by parts, we get

$$\int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx$$

$$= \frac{\pi}{2} + [\cos x]_0^{\pi/2} = \left(\frac{\pi}{2} - 1\right).$$

(ii) Integrating by parts, taking log (sin x) as the first function, we get

$$\int_0^{\pi} \cos 2x \log \sin x\,dx$$

$$= \left[(\log \sin x)\cdot\frac{\sin 2x}{2}\right]_0^{\pi} - \int_0^{\pi}\left(\cot x\cdot\frac{\sin 2x}{2}\right)dx$$

$$= 0 - \int_0^{\pi} \frac{\cos x}{\sin x}\cdot\frac{2\sin x\cos x}{2}\,dx = -\int_0^{\pi}\cos^2 x\,dx$$

$$= -\frac{1}{2}\int_0^{\pi} 2\cos^2 x\,dx = -\frac{1}{2}\int_0^{\pi}(1+\cos 2x)\,dx$$

$$= -\frac{1}{2}\cdot\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = -\frac{\pi}{2}.$$

(iii) Integrating by parts, taking (log x) as the first function, we get

$$\int_1^2 \frac{\log x}{x^2}\,dx = \int_1^2 (\log x)\cdot x^{-2}\,dx$$

$$= \left[(\log x)\left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx$$

$$= \left[-\frac{\log 2}{2} + \frac{\log 1}{1}\right] + \int_1^2 \frac{dx}{x^2}$$

$$= \frac{-\log 2}{2} - \left[\frac{1}{x}\right]_1^2 = \frac{-\log 2}{2} - \left\{\frac{1}{2} - 1\right\} = \left(\frac{1-\log 2}{2}\right).$$

(iv) $\int_0^{\pi/6}(2+3x^2)\cos 3x\,dx$

$$= 2\int_0^{\pi/6}\cos 3x\,dx + 3\int_0^{\pi/6}x^2\cos 3x\,dx$$

$$= 2\left[\frac{\sin 3x}{3}\right]_0^{\pi/6} + 3\left\{\left[x^2\left(\frac{\sin 3x}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}2x\left(\frac{\sin 3x}{3}\right)dx\right\}$$

[integrating by parts]

$$= \frac{2}{3} + \frac{\pi^2}{36} - 2\int_0^{\pi/6}x\sin 3x\,dx$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - 2\left\{\left[x\left(\frac{-\cos 3x}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}1\cdot\left(\frac{-\cos 3x}{3}\right)dx\right\}$$

[integrating by parts]

$$= \frac{2}{3} + \frac{\pi^2}{36} + \frac{2}{3}[x\cos 3x]_0^{\pi/6} - \frac{2}{3}\cdot\left[\frac{\sin 3x}{3}\right]_0^{\pi/6}$$

$$= \frac{2}{3} + \frac{\pi^2}{36} - \frac{2}{9} = \left(\frac{\pi^2}{36} + \frac{4}{9}\right) = \frac{1}{36}(\pi^2+16).$$

EXAMPLE 7 *Evaluate* $\int_1^2 \frac{dx}{x(1+x^2)}$.

SOLUTION Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{(1+x^2)}$.

Then, $1 \equiv A(1+x^2) + (Bx+C)x$. Putting $x = 0$, we get $A = 1$.

Comparing the coefficients of x^2, we get $A + B = 0$ or $B = -1$.

Comparing coefficients of x, we get $C = 0$.

$\therefore$ $$\frac{1}{x(1+x^2)} = \left[\frac{1}{x} - \frac{x}{1+x^2}\right]$$

So, $$\int_1^2 \frac{dx}{x(1+x^2)} = \int_1^2 \frac{dx}{x} - \frac{1}{2}\int_1^2 \frac{2x}{1+x^2}\,dx$$

$$= [\log x]_1^2 - \frac{1}{2}[\log(1+x^2)]_1^2$$

$$= \left[\frac{3}{2}(\log 2) - \frac{1}{2}(\log 5)\right].$$

EXAMPLE 8 *Evaluate* $\int_2^4 \frac{(x^2+x)}{\sqrt{2x+1}}\,dx.$

SOLUTION Integrating by parts, taking (x^2+x) as the first function and $\frac{1}{\sqrt{2x+1}}$ as the second function, we get

$$\int_2^4 \frac{(x^2+x)}{\sqrt{2x+1}}\,dx = [(x^2+x)\cdot\sqrt{2x+1}]_2^4 - \int_2^4 (2x+1)\cdot\sqrt{2x+1}\,dx$$

$$= (60-6\sqrt{5}) - \int_2^4 (2x+1)^{3/2}dx$$

$$= (60-6\sqrt{5}) - \frac{1}{5}\cdot[(2x+1)^{5/2}]_2^4$$

$$= (60-6\sqrt{5}) - \left(\frac{243}{5} - 5\sqrt{5}\right)$$

$$= \left(\frac{57}{5} - \sqrt{5}\right) = \sqrt{\frac{57-5\sqrt{5}}{5}}.$$

EXAMPLE 9 *Evaluate:* (i) $\int_0^{1/2} \frac{dx}{\sqrt{1-x}}$ (ii) $\int_0^1 \left(\frac{1-x}{1+x}\right)dx$

SOLUTION (i) $\int_0^{1/2} \frac{dx}{\sqrt{1-x}} = \int_0^{1/2} (1-x)^{-1/2}dx = \left[\frac{2\sqrt{1-x}}{-1}\right]_0^{1/2} = (2-\sqrt{2}).$

(ii) $\int_0^1 \left(\frac{1-x}{1+x}\right)dx = \int_0^1 \left(-1 + \frac{2}{x+1}\right)dx$ [on dividing $(-x+1)$ by $(x+1)$]

$$= [-x + 2\log|x+1|]_0^1 = [(2\log 2) - 1].$$

EXERCISE 16A

Very-Short-Answer Questions

Evaluate:

1. $\int_1^3 x^4 dx$
2. $\int_1^4 \sqrt{x}\,dx$
3. $\int_1^2 x^{-5}dx$
4. $\int_0^{16} x^{3/4}dx$
5. $\int_{-4}^{-1} \frac{dx}{x}$
6. $\int_1^4 \frac{dx}{\sqrt{x}}$

7. $\int_0^1 \frac{dx}{3\sqrt{x}}$

8. $\int_1^8 \frac{dx}{x^{2/3}}$

9. $\int_2^4 3\,dx$

10. $\int_0^1 \frac{dx}{(1+x^2)}$ **[CBSE 2008]**

11. $\int_0^\infty \frac{dx}{(1+x^2)}$

12. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

13. $\int_0^{\pi/6} \sec^2 x\,dx$

14. $\int_{-\pi/4}^{\pi/4} \text{cosec}^2 x\,dx$

15. $\int_{\pi/4}^{\pi/2} \cot^2 x\,dx$

16. $\int_0^{\pi/4} \tan^2 x\,dx$

17. $\int_0^{\pi/2} \sin^2 x\,dx$

18. $\int_0^{\pi/4} \cos^2 x\,dx$

19. $\int_0^{\pi/3} \tan\, x\,dx$

20. $\int_{\pi/6}^{\pi/4} \text{cosec}\, x\,dx$

21. $\int_0^{\pi/3} \cos^3 x\,dx$

22. $\int_0^{\pi/2} \sin^3 x\,dx$

23. $\int_{\pi/4}^{\pi/2} \frac{(1-3\cos x)}{\sin^2 x}\,dx$

24. $\int_0^{\pi/4} \sqrt{1+\cos 2x}\,dx$

25. $\int_0^{\pi/4} \sqrt{1-\sin\, 2x}\,dx$ **[CBSE 2004]**

26. $\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+\sin x)}$

27. $\int_0^{\pi/4} \frac{dx}{(1+\cos 2x)}$

28. $\int_{\pi/4}^{\pi/2} \frac{dx}{1-\cos 2x}$

29. $\int_0^{\pi/4} \sin\, 2x \sin\, 3x\,dx$

30. $\int_0^{\pi/6} \cos x \cos 2x\, dx$

31. $\int_0^{\pi} \sin\, 2x \cos\, 3x\,dx$

32. $\int_0^{\pi/2} \sqrt{1+\sin\, x}\,dx$

33. $\int_0^{\pi/2} \sqrt{1+\cos x}\,dx$

34. $\int_0^2 \frac{(x^4+1)}{(x^2+1)}\,dx$

35. $\int_1^2 \frac{dx}{(x+1)(x+2)}$

36. $\int_1^2 \frac{(x+3)}{x(x+2)}\,dx$

37. $\int_3^4 \frac{dx}{(x^2-4)}$

38. $\int_0^4 \frac{dx}{\sqrt{x^2+2x+3}}$

39. $\int_1^2 \frac{dx}{\sqrt{x^2+4x+3}}$

40. $\int_0^1 \frac{dx}{(1+x+2x^2)}$

Short-Answer Questions

Evaluate:

41. $\int_0^{\pi/2} (a\cos^2 x + b\sin^2 x)\,dx$

42. $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx$

43. $\int_0^{\pi/2} \cos^4 x\,dx$

44. $\int_0^a \frac{dx}{(ax+a^2-x^2)}$

45. $\int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}}$

46. $\int_0^1 \sqrt{x(1-x)}\,dx$

47. $\int_1^3 \frac{dx}{x^2(x+1)}$

48. $\int_1^2 \frac{dx}{x(1+2x)^2}$

49. $\int_0^1 x e^x dx$

50. $\int_0^{\pi/2} x^2 \cos x\,dx$

51. $\int_0^{\pi/4} x^2 \sin x\,dx$

52. $\int_0^{\pi/2} x^2 \cos 2x\,dx$ **[CBSE 2006C]**

53. $\int_0^{\pi/2} x^3 \sin 3x\,dx$

54. $\int_0^{\pi/2} x^2 \cos^2 x\,dx$

55. $\int_1^2 \log x\,dx$

56. $\int_1^3 \frac{\log x}{(1+x)^2}\,dx$

57. $\int_0^{e^2} \left\{\frac{1}{(\log x)} - \frac{1}{(\log x)^2}\right\} dx$

58. $\int_1^e e^x \left(\frac{1+x\log x}{x}\right) dx$

59. $\int_0^1 \frac{x\,e^x}{(1+x)^2}\,dx$

60. $\int_0^{\pi/2} 2\tan^3 x\,dx$ **[CBSE 2004]**

61. $\int_1^2 \frac{5x^2}{(x^2+4x+3)}\,dx$

ANSWERS (EXERCISE 16A)

1. $\frac{242}{5}$ **2.** $\frac{14}{3}$ **3.** $\frac{15}{64}$ **4.** $\frac{512}{7}$

5. $-\log 4$ **6.** 2 **7.** $\frac{3}{2}$ **8.** 3

9. 6 **10.** $\frac{\pi}{4}$ **11.** $\frac{\pi}{2}$ **12.** $\frac{\pi}{2}$

13. $\frac{1}{\sqrt{3}}$ **14.** -2 **15.** $\left(1-\frac{\pi}{4}\right)$ **16.** $\left(1-\frac{\pi}{4}\right)$

17. $\frac{\pi}{4}$ **18.** $\left(\frac{\pi}{8}+\frac{1}{4}\right)$ **19.** $\log 2$

20. $\log(\sqrt{2}-1)+\log(2+\sqrt{3})$ **21.** $\frac{3\sqrt{3}}{8}$ **22.** $\frac{2}{3}$

23. $(4-3\sqrt{2})$ **24.** 1 **25.** $(\sqrt{2}-1)$ **26.** 2

27. $\frac{1}{2}$ **28.** $\frac{1}{2}$ **29.** $\frac{3}{5\sqrt{2}}$ **30.** $\frac{5}{12}$

31. $\frac{-4}{5}$ **32.** 2 **33.** 2 **34.** $\left(\frac{2}{3}+2\tan^{-1}2\right)$

35. $(2\log 3-3\log 2)$ **36.** $\frac{1}{2}(\log 2+\log 3)$

37. $\frac{1}{4}(\log 5-\log 3)$ **38.** $\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$

39. $\log(4+\sqrt{15})-\log(3+\sqrt{8})$ **40.** $\frac{2}{\sqrt{7}}\left\{\tan^{-1}\frac{5}{\sqrt{7}}-\tan^{-1}\frac{1}{\sqrt{7}}\right\}$

41. $\frac{\pi}{4}(a+b)$ **42.** $\frac{-2}{\sqrt{3}}$ **43.** $\frac{3\pi}{16}$ **44.** $\frac{1}{\sqrt{5}a}\log\left\{\frac{7+3\sqrt{5}}{2}\right\}$

45. $\frac{\pi}{6}$ **46.** $\frac{\pi}{8}$ **47.** $\log 2-\log 3+\frac{2}{3}$

48. $\log 6-\log 5-\frac{2}{15}$ **49.** 1 **50.** $\left(\frac{\pi^2}{4}-2\right)$

51. $\left(\sqrt{2}+\frac{\pi}{2\sqrt{2}}-\frac{\pi^2}{16\sqrt{2}}-2\right)$ **52.** $\frac{-\pi}{4}$ **53.** $\left(\frac{2}{27}-\frac{\pi^2}{12}\right)$

54. $\left(\frac{\pi^3}{48}-\frac{\pi}{8}\right)$ **55.** $(2\log 2-1)$ **56.** $\frac{3}{4}\log 3-\log 2$ **57.** $\left(\frac{e^2}{2}-e\right)$

58. e^e **59.** $\left(\frac{e}{2}-1\right)$ **60.** $(1-\log 2)$

61. $5-\frac{5}{2}\left(9\log\frac{5}{4}-\log\frac{3}{2}\right)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16A)

15. $\cot^2 x = (\text{cosec}^2 x - 1)$.

16. $\tan^2 x = (\sec^2 x - 1)$.

17. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

18. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

19. $\int \tan x \, dx = \log \sec x$.

20. $\int \text{cosec}\, x \, dx = \log |\text{cosec}\, x - \cot x|$.

21. $\cos 3x = 4\cos^3 x - 3\cos x \Rightarrow \cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.

22. $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$.

23. $I = \int \left(\frac{1}{\sin^2 x} - \frac{3\cos x}{\sin^2 x} \right) dx = \int (\text{cosec}^2 x - 3\,\text{cosec}\, x \cot x)\, dx$.

25. $\sqrt{1 - \sin 2x} = (\cos^2 x + \sin^2 x - 2\sin x \cos x)^{1/2} = (\cos x - \sin x)$

$\left\{ \because \quad \cos x > \sin x \text{ in } \left[0, \frac{\pi}{4} \right] \right\}$.

26. $I = \int \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)} dx = \int \frac{(1-\sin x)}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$

$= \int (\sec^2 x - \sec x \tan x)\, dx$.

32. $\sqrt{1+\sin x} = \left\{ \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \right\}^{1/2} = \left(\cos\frac{x}{2} + \sin\frac{x}{2} \right)$.

34. $\frac{(x^4+1)}{(x^2+1)} = \left\{ x^2 - 1 + \frac{2}{(x^2+1)} \right\}$.

41. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

42. $I = \int (\tan^2 x + \cot^2 x + 2)\, dx = \int \{(\sec^2 x - 1) + (\text{cosec}^2 x - 1) + 2\}\, dx$

$= \int (\sec^2 x + \text{cosec}^2 x)\, dx$.

45. $I = \int \frac{dx}{\sqrt{\left\{ \frac{1}{4} - \left(x^2 - x + \frac{1}{4} \right) \right\}}} = \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} = \sin^{-1}(2x - 1)$.

46. $I = \int \sqrt{\left\{ \left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2 \right\}}\, dx$.

47. Let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. Then $Ax(x+1) + B(x+1) + Cx^2 \equiv 1$.

This, gives $A = -1, B = 1, C = 1$.

54. $I = \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{(1+x)^{-2}}\, dx$.

55. $I = \int \frac{dx}{(\log x)} - \int \frac{dx}{(\log x)^2} = I_1 - I_2$.

$I_1 = \int \{(\log x)^{-1} \cdot 1\}\, dx$. Integrate by parts.

58. $I = \int e^x \left(\log x + \frac{1}{x} \right) dx.$

59. $I = \int e^x \left\{ \frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right\} dx = e^x \cdot \frac{1}{(1+x)}.$

60. $I = 2\int \tan x \tan^2 x \, dx = 2\int \tan x \, (\sec^2 x - 1) \, dx.$

61. Let $\frac{5x^2}{(x^2 + 4x + 3)} = 5 + \frac{A}{(x+1)} + \frac{B}{(x+3)}.$

To Evaluate a Definite Integral by Substitution

In $\int_a^b f(x)\,dx$, when the variable x is converted into a new variable t by some relation then we put $x = a$ and $x = b$ in that relation to obtain the corresponding values of t, giving the lower limit and the upper limit respectively of the new integrand in t.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int_0^2 e^{x/2} dx$ *(ii)* $\int_2^4 \frac{x}{(x^2+1)} dx$

(iii) $\int_0^1 \cos^{-1} x \, dx$ *(iv)* $\int_0^1 \frac{(2x+3)}{(5x^2+1)} dx$

SOLUTION (i) Put $\frac{x}{2} = t$ so that $dx = 2\,dt$.

Also, $(x = 0 \Rightarrow t = 0)$ and $(x = 2 \Rightarrow t = 1)$.

$\therefore \quad \int_0^2 e^{x/2} dx = 2\int_0^1 e^t dt = 2[e^t]_0^1 = 2(e-1).$

(ii) Put $(x^2 + 1) = t$ so that $x\,dx = \frac{1}{2} dt$.

Also, $(x = 2 \Rightarrow t = 5)$ and $(x = 4 \Rightarrow t = 17)$.

$\therefore \int_2^4 \frac{x}{(x^2+1)} dx = \frac{1}{2}\int_5^{17} \frac{dt}{t} = \frac{1}{2}[\log |t|]_5^{17} = \frac{1}{2}(\log 17 - \log 5).$

(iii) Put $x = \cos t$ so that $dx = -\sin t \, dt$.

Also, $\left(x = 0 \Rightarrow t = \frac{\pi}{2}\right)$ and $(x = 1 \Rightarrow t = 0)$.

$$\therefore \int_0^1 \cos^{-1}x\,dx = -\int_{\pi/2}^0 \cos^{-1}(\cos t)\sin t\,dt = \int_0^{\pi/2} t\sin t\,dt$$

$$= [t(-\cos t)]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos t)\,dt$$

[integrating by parts]

$$= [\sin t]_0^{\pi/2} = 1.$$

(iv) Let $(2x+3) \equiv A\cdot\frac{d}{dx}(5x^2+1) + B.$

Then, $(2x+3) \equiv (10x)A + B.$

Comparing the coefficients of like powers of x, we get

$$10A = 2 \text{ or } A = \frac{1}{5} \text{ and } B = 3.$$

$$\therefore \quad (2x+3) = \frac{1}{5}(10x) + 3.$$

So, $$\int_0^1 \frac{(2x+3)}{(5x^2+1)}\,dx = \int_0^1 \frac{\frac{1}{5}(10x)+3}{(5x^2+1)}\,dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{(5x^2+1)}\,dx + 3\int_0^1 \frac{dx}{(5x^2+1)}$$

$$= \frac{1}{5}[\log|5x^2+1|]_0^1 + \frac{3}{5}\int_0^1 \frac{dx}{x^2+\left(\frac{1}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}\log 6 + \frac{3}{5}\cdot\sqrt{5}\left[\tan^{-1}\frac{x}{(1/\sqrt{5})}\right]_0^1$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}(\tan^{-1}\sqrt{5}).$$

EXAMPLE 2 *Evaluate:*

(i) $\int_1^3 \frac{\cos(\log x)}{x}\,dx$ *(ii)* $\int_0^{\pi/2} \sqrt{\cos\theta}\sin^3\theta\,d\theta$

(iii) $\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$ **[CBSE 2004]** *(iv)* $\int_0^{\pi/2} \frac{dx}{(1-2\sin x)}$

(v) $\int_0^{\pi/2} \frac{dx}{(3+2\cos x)}$ *(vi)* $\int_0^{\pi/2} \frac{dx}{(4\sin^2 x + 5\cos^2 x)}$

SOLUTION (i) Put $\log x = t$ so that $\frac{1}{x}dx = dt.$

Also, $(x=1 \Rightarrow t = \log 1 = 0)$ and $(x=3 \Rightarrow t = \log 3).$

$$\therefore \quad \int_1^3 \frac{\cos(\log x)}{x}\,dx = \int_0^{\log 3} \cos t\,dt = [\sin t]_0^{\log 3} = \sin(\log 3).$$

(ii) Put $\cos\theta = t$ so that $\sin\theta\,d\theta = -dt$.

Also, $(\theta = 0 \Rightarrow t = 1)$ and $\left(\theta = \frac{\pi}{2} \Rightarrow t = 0\right)$.

$$\therefore \quad \int_0^{\pi/2} \sqrt{\cos\theta}\sin^3\theta\,d\theta = \int_0^{\pi/2} \sqrt{\cos\theta}\cdot(1-\cos^2\theta)\sin\theta\,d\theta$$

$$= -\int_1^0 \sqrt{t}(1-t^2)\,dt = \int_0^1 (t^{1/2} - t^{5/2})\,dt$$

$$= \left[\frac{2}{3}t^{3/2} - \frac{2}{7}t^{7/2}\right]_0^1 = \left(\frac{2}{3} - \frac{2}{7}\right) = \frac{8}{21}.$$

(iii) Put $\sin x = t$ so that $\cos x\,dx = dt$.

Also, $(x = 0 \Rightarrow t = 0)$ and $\left(x = \frac{\pi}{2} \Rightarrow t = 1\right)$.

$$\therefore \quad \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx$$

$$= \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$= \int_0^1 \left[\frac{1}{(1+t)} - \frac{1}{(2+t)}\right] dt \qquad \text{[by partial fractions]}$$

$$= \int_0^1 \frac{dt}{(1+t)} - \int_0^1 \frac{dt}{(2+t)}$$

$$= [\log|1+t|]_0^1 - [\log|2+t|]_0^1$$

$$= [(\log 2 - \log 1) - (\log 3 - \log 2)] = (2\log 2) - (\log 3).$$

(iv) $$\int_0^{\pi/2} \frac{dx}{(1-2\sin x)} = \int_0^{\pi/2} \frac{dx}{1-2\left\{\frac{2\tan(x/2)}{1+\tan^2(x/2)}\right\}}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{[1+\tan^2(x/2) - 4\tan(x/2)]}\,dx$$

$$= 2\int_0^1 \frac{dt}{(1+t^2-4t)}, \text{ where } \tan\frac{x}{2} = t$$

$$\begin{bmatrix} x = 0 \Rightarrow t = 0 \\ x = \frac{\pi}{2} \Rightarrow t = 1 \end{bmatrix}$$

$$= 2\int_0^1 \frac{dt}{(t-2)^2 - (\sqrt{3})^2} = 2\cdot\frac{1}{2\sqrt{3}}\left[\log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right|\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[\log\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \log\frac{\sqrt{3}+2}{\sqrt{3}-2}\right].$$

(v) $$\int_0^{\pi/2} \frac{dx}{(3+2\cos x)} = \int_0^{\pi/2} \frac{dx}{3+2\cdot\left[\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right]}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{\tan^2(x/2)+5}\,dx$$

$$= 2\int_0^1 \frac{dt}{t^2+(\sqrt{5})^2}, \text{where } \tan\frac{x}{2} = t$$

$$\left[\begin{array}{l} x = 0 \Rightarrow t = 0 \\ x = \dfrac{\pi}{2} \Rightarrow t = 1 \end{array}\right]$$

$$= 2\cdot\frac{1}{\sqrt{5}}\left[\tan^{-1}\frac{t}{\sqrt{5}}\right]_0^1 = \frac{2}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}.$$

(vi) $$\int_0^{\pi/2} \frac{dx}{(4\sin^2 x + 5\cos^2 x)} = \int_0^{\pi/2} \frac{\sec^2 x}{(4\tan^2 x + 5)}\,dx$$

[dividing num. and denom. by $\cos^2 x$]

$$= \int_0^{\infty} \frac{dt}{(4t^2+5)}, \text{ where } \tan x = t$$

$$\left[\begin{array}{l} x = 0 \Rightarrow t = 0 \\ x = \dfrac{\pi}{2} \Rightarrow t = \infty \end{array}\right]$$

$$= \frac{1}{4}\int_0^{\infty} \frac{dt}{t^2+\left(\dfrac{\sqrt{5}}{2}\right)^2} = \frac{1}{4}\cdot\frac{2}{\sqrt{5}}\left[\tan^{-1}\frac{2t}{\sqrt{5}}\right]_0^{\infty}$$

$$= \frac{1}{2\sqrt{5}}[\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$= \frac{1}{2\sqrt{5}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4\sqrt{5}}.$$

EXAMPLE 3 *Evaluate:*

(i) $$\int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$$ *(ii)* $$\int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$$ **[CBSE 1996, '97]**

SOLUTION (i) Put $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$.

Clearly, $x = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \theta = (\pi/4)$.

$$\therefore \int_0^1 \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx = \int_0^{\pi/4} \frac{\theta\tan\theta}{\sec^3\theta}\cdot\sec^2\theta\,d\theta = \int_0^{\pi/4} \theta\sin\theta\,d\theta$$

$$= [-\theta\cos\theta]_0^{\pi/4} - \int_0^{\pi/4}(-\cos\theta)\,d\theta \qquad \text{[integrating by parts]}$$

$$= [-\theta\cos\theta]_0^{\pi/4} + [\sin\theta]_0^{\pi/4} = -\frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}$$

$$= \left(\frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{(4-\pi)}{4\sqrt{2}} = \frac{\sqrt{2}(4-\pi)}{8}.$$

(ii) Put $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$.

Clearly, $(x = 0 \Rightarrow 0 = 0)$ and $\left(x = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}\right)$.

$$\therefore \int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx = \int_0^{\pi/4} \frac{\theta}{\cos^3\theta}\cdot\cos\theta\,d\theta = \int_0^{\pi/4} \theta\sec^2\theta\,d\theta$$

$$= [\theta\tan\theta]_0^{\pi/4} - \int_0^{\pi/4} 1\cdot\tan\theta\,d\theta \quad \text{[integrating by parts]}$$

$$= \frac{\pi}{4} + [\log(\cos\theta)]_0^{\pi/4} = \frac{\pi}{4} + \log\left(\cos\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{\pi}{4} - \frac{1}{2}\log 2\right).$$

EXAMPLE 4 *Evaluate:* *(i)* $\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}\,dx$ *(ii)* $\int_0^{\pi/2} \frac{\cos x}{(1+\cos x+\sin x)}\,dx$

SOLUTION (i) $\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}dx = \int_0^{\pi/2} \frac{\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}dx$

$$= \int_0^{\pi/2} \frac{2\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}dx = 2\cdot\int_1^{\sqrt{2}} \frac{dt}{t^2} = \left[\frac{-2}{t}\right]_1^{\sqrt{2}} = \sqrt{2}(\sqrt{2}-1).$$

$$\left[\text{putting } \cos\frac{x}{2} + \sin\frac{x}{2} = t \text{ and } \frac{1}{2}\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)dx = dt;\right.$$

$$\left.\text{also, } x = 0 \Rightarrow t = 1 \text{ and } x = (\pi/2) \Rightarrow t = \sqrt{2}\right]$$

(ii) $\int_0^{\pi/2} \frac{\cos x}{(1+\cos x+\sin x)}dx = \int_0^{\pi/2} \frac{\cos x}{(1+\cos x)+\sin x}dx$

$$= \int_0^{\pi/2} \frac{\cos^2(x/2) - \sin^2(x/2)}{[2\cos^2(x/2) + 2\sin(x/2)\cos(x/2)]}\, dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1 - \tan^2(x/2)}{1 + \tan(x/2)}\, dx$$

[dividing num. and denom. by $\cos^2(x/2)$]

$$= \frac{1}{2}\int_0^{\pi/2} [1 - \tan(x/2)]\, dx = \frac{1}{2}\int_0^{\pi/2} dx - \frac{1}{2}\int_0^{\pi/2} \frac{\sin(x/2)}{\cos(x/2)}\, dx$$

$$= \frac{1}{2}\cdot[x]_0^{\pi/2} + [\log\cos(x/2)]_0^{\pi/2}$$

$$= \frac{\pi}{4} + \log\cos\frac{\pi}{4} = \frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{\pi}{4} - \frac{1}{2}\log 2\right).$$

EXAMPLE 5 *Evaluate* $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx.$ **[CBSE 2008, '11C]**

SOLUTION Put $x = a\cos\theta$ so that $dx = -a\sin\theta\, d\theta$.

Also, $(x = -a \Rightarrow \theta = \pi)$ and $(x = a \Rightarrow \theta = 0)$.

$$\therefore \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx = \int_{\pi}^{0} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cdot(-a\sin\theta)\, d\theta$$

$$= a\int_0^{\pi} \sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\cdot 2\sin(\theta/2)\cos(\theta/2)\, d\theta$$

$$= a\int_0^{\pi} 2\sin^2(\theta/2)\, d\theta = a\int_0^{\pi}(1-\cos\theta)\, d\theta$$

$$= a\int_0^{\pi} d\theta - a\int_0^{\pi}\cos\theta\, d\theta$$

$$= a\cdot[\theta]_0^{\pi} - a[\sin\theta]_0^{\pi} = a\pi.$$

EXAMPLE 6 *Evaluate* $\int_0^1 x\cdot\sqrt{\frac{1-x^2}{1+x^2}}\, dx.$ **[CBSE 2007]**

SOLUTION Put $x^2 = t$ and $x\, dx = \frac{1}{2}\, dt$. Then,

$[x = 0 \Rightarrow t = 0]$ and $[x = 1 \Rightarrow t = 1]$.

$$\therefore \quad I = \frac{1}{2}\cdot\int_0^1 \sqrt{\frac{1-t}{1+t}}\,dt$$

$$= \frac{1}{2}\cdot\int_0^1 \left\{\frac{\sqrt{1-t}}{\sqrt{1+t}}\times\frac{\sqrt{1-t}}{\sqrt{1-t}}\right\}dt = \frac{1}{2}\cdot\int_0^1 \frac{(1-t)}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\cdot\int_0^1 \frac{dt}{\sqrt{1-t^2}} - \frac{1}{2}\cdot\int_0^1 \frac{t}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\cdot[\sin^{-1}t]_0^1 + \frac{1}{4}\cdot\int_0^1 \frac{(-2t)}{\sqrt{1-t^2}}\,dt$$

$$= \frac{1}{2}\cdot[\sin^{-1}1 - \sin^{-1}0] + \frac{1}{4}\cdot\int_1^0 \frac{1}{\sqrt{u}}\,du, \text{ where } (1-t^2) = u$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right) - \frac{1}{4}\cdot\int_0^1 \frac{du}{\sqrt{u}} = \frac{\pi}{4} - \frac{1}{4}[2\sqrt{u}]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}[\sqrt{1}-\sqrt{0}] = \left(\frac{\pi}{4}-\frac{1}{2}\right).$$

EXAMPLE 7 *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(3\cos x + \sin x)}\,dx.$

SOLUTION Let $\cos x = A(3\cos x + \sin x) + B\cdot\frac{d}{dx}(3\cos x + \sin x)$

Then, $\cos x = A(3\cos x + \sin x) + B\cdot(-3\sin x + \cos x)$

Comparing the coefficients of $\cos x$, we get $3A + B = 1$.

Comparing the coefficients of $\sin x$, we get $A - 3B = 0$.

Solving $3A + B = 1$ and $A - 3B = 0$, we get $A = \frac{3}{10}$ and $B = \frac{1}{10}$.

$$\therefore \quad \cos x = \frac{3}{10}(3\cos x + \sin x) + \frac{1}{10}(-3\sin x + \cos x).$$

So, $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(3\cos x + \sin x)}\,dx$

$$= \frac{3}{10}\int_0^{\pi/2} \frac{(3\cos x + \sin x)}{(3\cos x + \sin x)}\,dx + \frac{1}{10}\int_0^{\pi/2} \frac{(-3\sin x + \cos x)}{(3\cos x + \sin x)}\,dx$$

$$= \frac{3}{10}\int_0^{\pi/2} dx + \frac{1}{10}\int_0^{\pi/2} \frac{(-3\sin x + \cos x)}{(3\cos x + \sin x)}\,dx$$

$$= \frac{3}{10}\cdot[x]_0^{\pi/2} + \frac{1}{10}\cdot[\log|3\cos x + \sin x|]_0^{\pi/2} = \left(\frac{3\pi}{20} - \frac{\log 3}{10}\right).$$

EXAMPLE 8 *Evaluate* $\int_0^{\pi/2} \{\sqrt{\tan x} + \sqrt{\cot x}\}\, dx.$ **[CBSE 2003, '12]**

SOLUTION We have

$$I = \int_0^{\pi/2} \left\{\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right\} dx = \int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{\sin x \cos x}}\, dx$$

$$= \sqrt{2} \cdot \int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{2 \sin x \cos x}}\, dx = \sqrt{2} \cdot \int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx$$

Put $(\sin x - \cos x) = t$ and $(\cos x + \sin x)\, dx = dt$.

Also, $[x = 0 \Rightarrow t = -1]$ and $\left[x = \frac{\pi}{2} \Rightarrow t = 1\right]$.

$$\therefore \quad I = \sqrt{2} \cdot \int_{-1}^{1} \frac{dt}{\sqrt{1 - t^2}} = \sqrt{2}[\sin^{-1} t]_{-1}^{1}$$

$$= \sqrt{2}\{\sin^{-1}(1) - \sin^{-1}(-1)\} = \sqrt{2}\{2 \sin^{-1}(1)\}$$

$$= \left(\sqrt{2} \times 2 \times \frac{\pi}{2}\right) = \sqrt{2}\pi.$$

EXERCISE 16B

Very-Short-Answer Questions

Evaluate the following integrals:

1. $\int_0^1 \frac{dx}{(2x - 3)}$
2. $\int_0^1 \frac{2x}{(1 + x^2)}\, dx$ **[CBSE 2008]**
3. $\int_1^2 \frac{3x}{(9x^2 - 1)}\, dx$
4. $\int_0^1 \frac{\tan^{-1} x}{(1 + x^2)}\, dx$
5. $\int_0^1 \frac{e^x}{(1 + e^{2x})}\, dx$ **[CBSE 1997]**
6. $\int_0^1 \frac{2x}{(1 + x^4)}\, dx$
7. $\int_0^1 x e^{x^2}\, dx$
8. $\int_1^2 \frac{e^{1/x}}{x^2}\, dx$
9. $\int_0^{\pi/6} \frac{\cos x}{(3 + 4 \sin x)}\, dx$
10. $\int_0^{\pi/2} \frac{\sin x}{(1 + \cos^2 x)}\, dx$ **[CBSE 1997]**
11. $\int_0^1 \frac{dx}{(e^x + e^{-x})}$
12. $\int_{1/e}^{e} \frac{dx}{x(\log x)^{1/3}}$

Short-Answer Questions

Evaluate the following integrals:

13. $\int_0^1 \frac{\sqrt{\tan^{-1}x}}{(1+x^2)}\,dx$

14. $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}}\,dx$

15. $\int_0^{\pi/2} \sqrt{\sin x}\cdot\cos^5 x\,dx$

16. $\int_0^{\pi/2} \frac{\sin x\cos x}{(1+\sin^4 x)}\,dx$

17. $\int_0^a \sqrt{a^2-x^2}\,dx$

18. $\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$

19. $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$

20. $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\,dx$

21. $\int_0^2 x\sqrt{2-x}\,dx$

22. $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ **[CBSE 2002]**

23. $\int_0^{\pi/2} \sqrt{1+\cos x}\,dx$

24. $\int_0^{\pi/2} \sqrt{1+\sin x}\,dx$

25. $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)}$ **[CBSE 2003C]**

26. $\int_0^{\pi/2} \frac{dx}{(1+\cos^2 x)}$

27. $\int_0^{\pi/2} \frac{dx}{(4+9\cos^2 x)}$

28. $\int_0^{\pi/2} \frac{dx}{(5+4\sin x)}$

29. $\int_0^{\pi} \frac{dx}{(6-\cos x)}$

30. $\int_0^{\pi} \frac{dx}{(5+4\cos x)}$ **[CBSE 2005]**

31. $\int_0^{\pi/2} \frac{dx}{(\cos x+2\sin x)}$

32. $\int_0^{\pi} \frac{dx}{(3+2\sin x+\cos x)}$ **[CBSE 2004]**

33. $\int_0^{\pi/4} \frac{\tan^3 x}{(1+\cos 2x)}\,dx$

34. $\int_0^{\pi/2} \frac{\sin x\cos x}{(\cos^2 x+3\cos x+2)}\,dx$

35. $\int_0^{\pi/2} \frac{\sin 2x}{(\sin^4 x+\cos^4 x)}\,dx$ **[CBSE 2003]**

36. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$

37. $\int_0^1 (\cos^{-1}x)^2\,dx$

38. $\int_0^1 x(\tan^{-1}x)^2\,dx$

39. $\int_0^1 \sin^{-1}\sqrt{x}\,dx$

40. $\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ **[CBSE 2008]**

41. $\int_0^9 \frac{dx}{(1+\sqrt{x})}$

42. $\int_0^1 x^3\sqrt{1+3x^4}\,dx$

43. $\int_0^1 \frac{(1-x^2)}{(1+x^2)^2}\,dx$

44. $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$

45. $\int_0^{\pi/2} (\sqrt{\tan x}+\sqrt{\cot x})\,dx$ **[CBSE 2002, '03, '05, '12]**

46. $\int_2^3 \frac{(2-x)}{\sqrt{5x-6-x^2}}\,dx$

47. $\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^3}\,d\theta$

48. $\int_0^{(\pi/2)^{1/3}} x^2 \sin x^3 dx$

49. $\int_1^2 \frac{dx}{x(1+\log x)^2}$ **[CBSE 2003]**

50. $\int_{\pi/6}^{\pi/2} \frac{\text{cosec}\,x\cot x}{1+\text{cosec}^2 x}\,dx$

ANSWERS (EXERCISE 16B)

1. $-\frac{1}{2}\log 3$ **2.** $\log 2$ **3.** $\frac{1}{6}(\log 35-\log 8)$

4. $\frac{\pi^2}{32}$ **5.** $\left(\tan^{-1}e-\frac{\pi}{4}\right)$ **6.** $\frac{\pi}{4}$

7. $\left(\frac{e-1}{2}\right)$ **8.** $(e-\sqrt{e})$ **9.** $\frac{1}{4}(\log 5-\log 3)$

10. $\frac{\pi}{4}$ **11.** $\left(\tan^{-1}e-\frac{\pi}{4}\right)$ **12.** 0

13. $\frac{1}{12}\pi^{3/2}$ **14.** $2(\sqrt{2}-1)$ **15.** $\frac{64}{231}$ **16.** $\frac{\pi}{8}$

17. $\frac{\pi a^2}{4}$ **18.** $\frac{\pi}{2}$ **19.** $\frac{3\pi a^4}{16}$ **20.** $a(\sqrt{2}-1)$

21. $\frac{16\sqrt{2}}{15}$ **22.** $\left(\frac{\pi}{2}-\log 2\right)$ **23.** 2 **24.** 2

25. $\frac{\pi}{2ab}$ **26.** $\frac{\pi}{2\sqrt{2}}$ **27.** $\frac{\pi}{4\sqrt{13}}$ **28.** $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

29. $\frac{\pi}{\sqrt{35}}$ **30.** $\frac{\pi}{3}$ **31.** $\frac{1}{\sqrt{5}}\log\left|\frac{3-\sqrt{5}}{2}\right|$

32. $\frac{\pi}{4}$ **33.** $\frac{1}{8}$ **34.** $(\log 9 - \log 8)$

35. $\frac{\pi}{2}$ **36.** $\frac{3}{2}$ **37.** $(\pi - 2)$ **38.** $\frac{\pi}{4}\left(\frac{\pi}{4} - 1\right)$

39. $\frac{\pi}{4}$ **40.** $a\left(\frac{\pi}{2} - 1\right)$ **41.** $(6 - 4\log 2)$ **42.** $\frac{7}{18}$

43. $\frac{1}{2}$ **44.** $\frac{1}{\sqrt{3}}$ **45.** $\pi\sqrt{2}$ **46.** $-\frac{\pi}{2}$

47. $\frac{2}{\left(\cos\frac{\pi}{8} + \sin\frac{\pi}{8}\right)} - \sqrt{2}$ **48.** $\frac{1}{3}$ **49.** $\frac{\log 2}{(1 + \log 2)}$

50. $\tan^{-1}\frac{1}{3}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16B)

5. Put $e^x = t$.

8. Putting $\frac{1}{x} = t$, we get $I = -\int_1^{1/2} e^t dt = \int_{1/2}^{1} e^t dt = [e^t]_{1/2}^1$.

9. Putting $3 + 4\sin x = t$, we get $I = \frac{1}{4} \cdot \int_3^5 \frac{dt}{t}$.

10. Put $\cos x = t$.

11. $I = \int_0^1 \frac{e^x}{(1 + e^{2x})}\, dx$.

15. $I = \int_0^{\pi/2} \sqrt{\sin x} \cdot (1 - \sin^2 x)^2 \cos x\, dx$. Now, put $\sin x = t$.

16. Put $\sin^2 x = t$.

18. Put $x = \sqrt{2}\sin t$.

20. Put $x = a\tan\theta$.

21. Put $(x + 2) = t^2$.

22. Put $x = \tan t$ and integrate by parts.

25. Divide num. and denom. by $\cos^2 x$ and put $\tan x = t$.

32. Write $\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$, $\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$ and put $\tan\frac{x}{2} = t$.

33. Using $\cos 2x = (2\cos^2 x - 1)$, we get $I = \int_0^{\pi/4} \tan^3 x \sec^2 x\, dx$.

Now, put $\tan x = t$.

34. Put $\cos x = t$.

35. Divide num. and denom. by $\cos^4 x$ and put $\tan^2 x = t$.

36. $I = \int_{\pi/3}^{\pi/2} \frac{\cos(x/2)}{\sin^5(x/2)}\, dx$. Put $\sin\frac{x}{2} = t$.

37. Put $x = \cos t$.

38. Integrate by parts with x as the second function.

39. Put $x = \sin^2 t$. 40. Put $x = a\tan^2\theta$. 41. Put $x = t^2$.

42. Put $(1 + 3x^4) = t$. 43. Put $x = \tan t$. 44. Put $(x + 1) = \frac{1}{t}$.

45. $I = \int_0^{\pi/2} \frac{\sqrt{2}(\sin x + \cos x)}{\sqrt{2\sin x\cos x}}\,dx = \sqrt{2}\cdot\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{1-(1-\sin 2x)}}\,dx$

$= \sqrt{2}\cdot\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin x - \cos x)^2}}\,dx.$ Put $(\sin x - \cos x) = t$.

46. Let $(2 - x) = A\cdot\frac{d}{dx}(5x - 6 - x^2) + B$.

47. $I = \int_{\pi/4}^{\pi/2} \frac{\left[\cos\frac{0}{2} - \sin\frac{0}{2}\right]}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^{1/2}}\,dx.$ Now, put $\cos\frac{\theta}{2} + \sin\frac{\theta}{2} = t$.

Properties of Definite Integrals

THEOREM 1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

PROOF Let $\int f(x)\,dx = F(x)$. Then, $\int f(t)\,dt = F(t)$.

$\therefore$ $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

And, $\int_a^b f(t)\,dt = [F(t)]_a^b = F(b) - F(a)$.

Hence, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

THEOREM 2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

PROOF Let $\int f(x)\,dx = F(x)$.

Then, $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

And, $-\int_b^a f(x)\,dx = -[F(x)]_b^a = -[F(a) - F(b)] = F(b) - F(a)$.

Hence, $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

THEOREM 3 $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad where\ \ a < c < b.$

PROOF Let $\int f(x)\,dx = F(x)$. Then, $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = [F(x)]_a^c + [F(x)]_c^b = \{F(c) - F(a)\} + \{F(b) - F(c)\}$$

$$= F(b) - F(a) = [F(x)]_a^b = \int_a^b f(x)\,dx.$$

Hence, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

REMARK If $a < c_1 < c_2 < \ldots < c_n < b$ then

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)\,dx.$$

THEOREM 4 $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$ **[CBSE 2002C, '05C, '06]**

PROOF In the RHS integral, put $(a - x) = t$ so that $dx = -dt$.

Now, when $x = 0$ we have $t = a$.

And, when $x = a$ we have $t = 0$.

$$\therefore \quad \int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx.$$

Hence, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

THEOREM 5 $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx.$

PROOF Putting $a + b - x = t$, we get $dx = -dt$.

Now, $x = a \Rightarrow t = b$.

And, $x = b \Rightarrow t = a$.

$$\therefore \quad \int_a^b f(a+b-x)\,dx = -\int_b^a f(t)\,dt = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx.$$

Hence, $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx.$

THEOREM 6 $\int_a^b \{f(x) + g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

PROOF Let $\int f(x)\,dx = F(x)$ and $\int g(x)\,dx = G(x).$

Then, $\int [f(x)+g(x)]dx = \int f(x)\,dx + \int g(x)\,dx = F(x)+G(x).$

$\therefore \quad \int_a^b \{f(x)+g(x)\}\,dx = [F(x)+G(x)]_a^b$

$= [F(b)+G(b)] - [F(a)+G(a)]$

$= [F(b)-F(a)] + [G(b)-G(a)]$

$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Hence, $\int_a^b \{f(x)+g(x)\}\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

THEOREM 7 $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, \text{ when } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)\,dx, \text{ when } f(x) \text{ is an even function.} \end{cases}$ **[CBSE 2005C]**

PROOF We have $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$ … (i)

In the first integral on the RHS of (i), put $x = -t$ so that $dx = -dt$.

When $x = -a$, we have $t = a$. And, when $x = 0$, we have $t = 0$.

$\therefore \quad \int_{-a}^{0} f(x)\,dx = -\int_a^0 f(-t)\,dt = \int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx.$

Thus, $\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dx$ … (ii)

Using (ii) in (i), we have

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a [f(-x)+f(x)]\,dx$$

$$= \begin{cases} 0, \text{ when } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, \text{ when } f(x) \text{ is even.} \end{cases}$$

$$\left[\because \begin{array}{l} f(x) \text{ is odd} \Rightarrow f(-x) = -f(x) \\ f(x) \text{ is even} \Rightarrow f(-x) = f(x) \end{array}\right]$$

THEOREM 8 $\int_0^{2a} f(x)\,dx = \begin{cases} 0, \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x). \end{cases}$

PROOF We have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$ … (i)

Now, in the second integral on the RHS of (i), put $x = 2a - t$ so that $dx = -dt$.

When $x = a$, we have $t = a$. When $x = 2a$, we have $t = 0$.

$$\therefore \quad \int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt$$

$$= \int_0^a f(2a-x)\,dx.$$

Thus, $$\int_a^{2a} f(x)\,dx = \int_0^a f(2a-x)\,dx \qquad \ldots \text{(ii)}$$

Using (ii) in (i), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \int_0^a \{f(x) + f(2a-x)\}\,dx$$

$$= \begin{cases} 0, \text{ when } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ when } f(2a-x) = f(x). \end{cases}$$

Hence, $$\int_0^{2a} f(x)\,dx = \begin{cases} 0, \text{ when } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ when } f(2a-x) = f(x). \end{cases}$$

SOLVED EXAMPLES

EXAMPLE 1 *Prove that* $\displaystyle\int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx = \frac{\pi}{4}$. **[CBSE 2002C, '07C]**

SOLUTION Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx \qquad \ldots \text{(i)}$

Using the result $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in (i), we get

$$I = \int_0^{\pi/2} \frac{\sin[(\pi/2) - x]}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}\,dx$$

or $$I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)}\,dx = \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\,dx \qquad \ldots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx + \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\,dx$$

$$= \int_0^{\pi/2} \frac{(\sin x + \cos x)}{(\sin x + \cos x)}\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

$$\therefore \quad I = \frac{\pi}{4}, \text{ i.e., } \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx = \frac{\pi}{4}.$$

EXAMPLE 2 *Prove that* $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \frac{\pi}{4}.$

SOLUTION Let $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx$ … (i)

Using the result $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in (i), we get

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos[(\pi/2) - x]}}{\sqrt{\sin[(\pi/2) - x]} + \sqrt{\cos[(\pi/2) - x]}}\,dx$$

or $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx$$

$$= \int_0^{\pi/2} \frac{(\sqrt{\sin x} + \sqrt{\cos x})}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

$$\therefore \quad I = \frac{\pi}{4}, \text{ i.e., } \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx = \frac{\pi}{4}.$$

EXAMPLE 3 *Evaluate:* (i) $\displaystyle\int_0^1 x(1-x)^n\,dx$ (ii) $\displaystyle\int_0^1 x(1-x)^{3/2}\,dx$

SOLUTION (i) We have $\displaystyle\int_0^1 x(1-x)^n\,dx = \int_0^1 (1-x)[1-(1-x)]^n\,dx$

$$[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx]$$

$$= \int_0^1 (1-x)x^n\,dx = \int_0^1 x^n\,dx - \int_0^1 x^{n+1}\,dx$$

$$= \left[\frac{x^{n+1}}{(n+1)}\right]_0^1 - \left[\frac{x^{n+2}}{n+2}\right]_0^1 = \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{(n+1)(n+2)}.$$

(ii) We have $\int_0^1 x(1-x)^{3/2}dx = \int_0^1 (1-x)[1-(1-x)]^{3/2}dx$

$$[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx]$$

$$= \int_0^1 (1-x)x^{3/2}dx = \int_0^1 x^{3/2}dx - \int_0^1 x^{5/2}dx$$

$$= \left[\frac{2}{5}x^{5/2}\right]_0^1 - \left[\frac{2}{7}x^{7/2}\right]_0^1 = \left(\frac{2}{5} - \frac{2}{7}\right) = \frac{4}{35}.$$

EXAMPLE 4 *Show that* $\int_0^{\pi/2} \log(\tan x)\,dx = 0.$

SOLUTION Let $I = \int_0^{\pi/2} \log(\tan x)\,dx$ … (i)

Then, $I = \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right]dx \quad [\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx]$

or $I = \int_0^{\pi/2} \log(\cot x)\,dx = \int_0^{\pi/2} \log\left(\frac{1}{\tan x}\right)dx = -\int_0^{\pi/2} \log \tan x\,dx = -I.$

$\therefore \quad I = -I$ or $2I = 0$ or $I = 0.$

Hence, $\int_0^{\pi/2} \log(\tan x)\,dx = 0.$

EXAMPLE 5 *Prove that* $\int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}(\log 2).$ **[CBSE 2007, '11, '13C]**

SOLUTION Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ … (i)

Then, $I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4} - x\right)\right]dx \quad [\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx]$

or $I = \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$

or $$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx$$

or $$I = \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\, dx$$

or $$I = (\log 2) \cdot \int_0^{\pi/4} dx - \int_0^{\pi/4} \log(1+\tan x)\, dx \qquad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = (\log 2) \int_0^{\pi/4} dx = (\log 2) \cdot [x]_0^{\pi/4} = \frac{\pi}{4} (\log 2).$$

$$\therefore \quad I = \frac{\pi}{8} (\log 2), \text{ i.e., } \int_0^{\pi/4} \log(1+\tan x)\, dx = \frac{\pi}{8} (\log 2).$$

EXAMPLE 6 *Prove that* $$\int_0^{\pi/2} \log(\sin x)\, dx = \int_0^{\pi/2} \log(\cos x)\, dx = -\frac{\pi}{2} (\log 2).$$

[CBSE 2004, '07C, '08]

SOLUTION Let $$I = \int_0^{\pi/2} \log(\sin x)\, dx \qquad \text{... (i)}$$

Then, $$I = \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

or $$I = \int_0^{\pi/2} \log(\cos x)\, dx \qquad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)]\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log(\sin 2x)\, dx - \int_0^{\pi/2} (\log 2)\, dx = \frac{1}{2} \int_0^{\pi} \log \sin t\, dt - (\log 2) \cdot \int_0^{\pi/2} dx$$

[putting $2x = t$ in the 1st integral]

$$= \frac{1}{2} \int_0^{\pi} \log \sin t\, dt - (\log 2) \cdot [x]_0^{\pi/2}$$

$$= \left(\frac{1}{2} \times 2\right) \cdot \int_0^{\pi/2} \log \sin t\, dt - \frac{\pi}{2} (\log 2)$$

$$= \int_0^{\pi/2} \log(\sin x)\, dx - \frac{\pi}{2} (\log 2).$$

$$\therefore \quad 2I = I - \frac{\pi}{2}(\log 2) \quad \text{or} \quad I = -\frac{\pi}{2}(\log 2).$$

$$\therefore \quad \int_0^{\pi/2} \log(\sin x)\,dx = \int_0^{\pi/2} \log(\cos x)\,dx = -\frac{\pi}{2}(\log 2).$$

EXAMPLE 7 *Prove that:* (a) $\int_0^{\pi/2} \sin 2x \log(\tan x)\,dx = 0$ **[CBSE 2003C, '06C]**

(b) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0$

SOLUTION (a) Let $I = \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx$ … (i)

Then, $I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

or $I = \int_0^{\pi/2} \sin 2x \log(\cot x)\,dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} [\sin 2x \log(\tan x) + \sin 2x \log(\cot x)]\,dx$$

$$= \int_0^{\pi/2} \sin 2x \{\log(\tan x) + \log(\cot x)\}\,dx$$

$$= \int_0^{\pi/2} \sin 2x \cdot \log(\tan x \cdot \cot x)\,dx = \int_0^{\pi/2} \sin 2x \cdot \log(1)\,dx = 0$$

$$[\because \quad \log 1 = 0]$$

$$\therefore \quad I = 0 \Rightarrow \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx = 0.$$

(b) Put $x = \cos^2 t$ so that $dx = -\sin 2t\,dt$.

When $x = 0$, we have $\cos t = 0$ or $t = \frac{\pi}{2}$.

When $x = 1$, we have $\cos t = 1$ or $t = 0$.

$$\therefore \quad \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = -\int_{\pi/2}^0 \log(\tan^2 t) \cdot \sin 2t\,dt$$

$$= 2\int_0^{\pi/2} \sin 2t \cdot \log(\tan t)\,dt = 0 \quad \text{[from (a)].}$$

EXAMPLE 8 *Prove that:* (a) $\int_0^{\pi/2} \frac{\sin^2 x}{(\sin x + \cos x)} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$ **[CBSE 2012, '14C]**

(b) $\int_0^{\pi/2} \frac{\sin^2 x}{(1+\sin x \cos x)} dx = \frac{\pi}{3\sqrt{3}}$.

SOLUTION (a) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{(\sin x + \cos x)} dx$ ... (i)

Then, $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\left[\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right]} dx$

or $I = \int_0^{\pi/2} \frac{\cos^2 x}{(\cos x + \sin x)} dx = \int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)} dx$ ... (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(\sin x + \cos x)} dx = \int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)}$$

$$= \int_0^{\pi/2} \frac{dx}{\left[\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right]}$$

$$= \int_0^{\pi/2} \frac{\sec^2(x/2)}{[1-\tan^2(x/2) + 2\tan(x/2)]} dx$$

$$= 2\int_0^1 \frac{dt}{(1-t^2+2t)}, \quad \text{where } \tan\frac{x}{2} = t$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1\right]$$

$$= 2\int_0^1 \frac{dt}{2-(t-1)^2} = 2\int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{2}} \left\{\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right\}_0^1 = \frac{1}{\sqrt{2}}\left[0 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{1}{\sqrt{2}} \log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)^2 = \frac{1}{\sqrt{2}} \times 2\log(\sqrt{2}+1)$$

$$= \sqrt{2}\log(\sqrt{2}+1)$$

$\therefore \quad I = \frac{1}{2}\sqrt{2}\log(\sqrt{2}+1) = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$

(b) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{(1+\sin x \cos x)} dx$ … (i)

Then, $I = \int_0^{\pi/2} \frac{\sin^2[(\pi/2)-x]}{1+\sin[(\pi/2)-x]\cos[(\pi/2)-x]} dx$

or $I = \int_0^{\pi/2} \frac{\cos^2 x}{(1+\sin x \cos x)} dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(1+\sin x \cos x)} dx = \int_0^{\pi/2} \frac{dx}{(1+\sin x \cos x)}$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(\sec^2 x + \tan x)} dx$$

[dividing num. and denom. by $\cos^2 x$]

$$= \int_0^{\pi/2} \frac{\sec^2 x}{(1+\tan^2 x + \tan x)} dx = \int_0^{\infty} \frac{dt}{(t^2+t+1)}, \text{ where } \tan x = t$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = \infty\right]$$

$$= \int_0^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \left[\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{2}{\sqrt{3}} \cdot \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}.$$

Hence, $I = \frac{\pi}{3\sqrt{3}}$.

EXAMPLE 9 *Prove that* $\int_0^{\pi} \frac{x \tan x}{(\sec x + \tan x)} dx = \pi\left(\frac{\pi}{2} - 1\right)$. **[CBSE 2008, '10]**

SOLUTION Let $I = \int_0^{\pi} \frac{x \tan x}{(\sec x + \tan x)} dx$ … (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{[\sec(\pi - x) + \tan(\pi - x)]} dx$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

or $I = \int_0^{\pi} \frac{(\pi - x)\tan x}{(\sec x + \tan x)} dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\tan x}{(\sec x + \tan x)}dx = \pi\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{(\sec^2 x - \tan^2 x)}dx$$

$$= \pi\cdot\left[\int_0^{\pi} \sec x \tan x\, dx - \int_0^{\pi} \tan^2 x\, dx\right]$$

$$= \pi\cdot\left\{[\sec x]_0^{\pi} - \int_0^{\pi} (\sec^2 x - 1)dx\right\}$$

$$= \pi\cdot\{-2 - [\tan x]_0^{\pi} + [x]_0^{\pi}\} = \pi(\pi - 2).$$

$$\therefore \quad I = \pi\left(\frac{\pi}{2} - 1\right) \quad \text{i.e.,} \quad \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}dx = \pi\left(\frac{\pi}{2} - 1\right).$$

EXAMPLE 10 *Evaluate* $\int_0^{\pi} \frac{x}{(1+\sin x)}dx.$ **[CBSE 2010, '11, '12C]**

SOLUTION Let $I = \int_0^{\pi} \frac{x}{(1+\sin x)}dx$ … (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)}dx = \int_0^{\pi} \frac{(\pi - x)}{(1+\sin x)}dx$ …(ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{dx}{(1+\sin x)} = \pi\cdot\int_0^{\pi} \frac{1}{(1+\sin x)} \times \frac{(1-\sin x)}{(1-\sin x)}dx$$

$$\text{or} \quad 2I = \pi\int_0^{\pi} \left(\frac{1-\sin x}{\cos^2 x}\right)dx = \pi\cdot\left[\int_0^{\pi} \sec^2 x\, dx - \int_0^{\pi} \sec x \tan x\, dx\right]$$

$$= \pi\cdot\{[\tan x]_0^{\pi} - [\sec x]_0^{\pi}\} = 2\pi$$

$$\therefore \quad I = \pi, \quad \text{i.e.,} \quad \int_0^{\pi} \frac{x}{(1+\sin x)}dx = \pi.$$

EXAMPLE 11 *Evaluate* $\int_0^{\pi} \frac{x \sin x}{(1+\cos^2 x)}dx.$ **[CBSE 2009C, '11C, '12, '14]**

SOLUTION Let $I = \int_0^{\pi} \frac{x \sin x}{(1+\cos^2 x)}dx$ … (i)

Then, $I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}dx$

or $I = \int_0^{\pi} \frac{(\pi - x)\sin x}{(1+\cos^2 x)}dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\sin x}{(1+\cos^2 x)}\,dx = -\pi\int_1^{-1} \frac{dt}{(1+t^2)}, \text{ where } \cos x = t$$

$$[x = 0 \Rightarrow t = 1 \text{ and } x = \pi \Rightarrow t = -1]$$

$$= \pi\int_{-1}^{1} \frac{dt}{(1+t^2)} = \pi[\tan^{-1} t]_{-1}^{1}$$

$$= \pi[\tan^{-1} 1 - \tan^{-1}(-1)] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}.$$

$$\therefore \quad I = \frac{\pi^2}{4}.$$

EXAMPLE 12 *Evaluate* $\int_0^{\pi/2} \frac{x}{(\sin x + \cos x)}\,dx.$ **[CBSE 2003C, '08]**

SOLUTION Let $I = \int_0^{\pi/2} \frac{x}{(\sin x + \cos x)}\,dx$ … (i)

Then, $I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{\sin[(\pi/2) - x] + \cos[(\pi/2) - x]}\,dx$

or $I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{(\cos x + \sin x)}\,dx = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)}{(\sin x + \cos x)}\,dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)}$$

$$\therefore \quad I = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)} = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\left[\frac{2\tan(x/2)}{1+\tan^2(x/2)} + \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right]}$$

$$= \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2(x/2)}{1 - \tan^2(x/2) + 2\tan(x/2)}\,dx$$

$$= \frac{\pi}{4}\int_0^{1} \frac{2dt}{(1 - t^2 + 2t)}, \quad \text{where } t = \tan\frac{x}{2}$$

$$\left[x = 0 \Rightarrow t = 0 \text{ and } x = \frac{\pi}{2} \Rightarrow t = 1\right]$$

$$= \frac{\pi}{2}\int_0^{1} \frac{dt}{[(\sqrt{2})^2 - (t-1)^2]}\,dt$$

$$= \frac{\pi}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2} + (t-1)}{\sqrt{2} - (t-1)}\right|_0^1 = \frac{\pi}{4\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|.$$

EXAMPLE 13 *Prove that* $\int_0^{\pi/2}(2\log \sin x - \log \sin 2x)\,dx = -\frac{\pi}{2}(\log 2)$. **[CBSE 2009]**

SOLUTION Let $I = \int_0^{\pi/2}(2\log \sin x - \log \sin 2x)\,dx.$ … (i)

Then, $I = \int_0^{\pi/2}\left[2\log \sin\left(\frac{\pi}{2}-x\right) - \log \sin 2\left(\frac{\pi}{2}-x\right)\right]dx$

or $I = \int_0^{\pi/2}(2\log \cos x - \log \sin 2x)\,dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2}[2(\log \sin x + \log \cos x) - 2\log \sin 2x]dx$$

or $$I = \int_0^{\pi/2}[\log (\sin x \cos x) - \log \sin 2x]dx$$

$$= \int_0^{\pi/2}\left[\log\left(\frac{\sin 2x}{2}\right) - \log \sin 2x\right]dx$$

$$= \int_0^{\pi/2}(\log \sin 2x - \log 2 - \log \sin 2x)\,dx$$

$$= -\log 2 \cdot \int_0^{\pi/2} dx = (-\log 2)\cdot[x]_0^{\pi/2} = -\frac{\pi}{2}(\log 2).$$

EXAMPLE 14 *Evaluate* $\int_0^{\pi}\frac{x}{(a^2\cos^2 x + b^2 \sin^2 x)}dx.$ **[CBSE 2003C, '09]**

SOLUTION Let $I = \int_0^{\pi}\frac{x}{(a^2\cos^2 x + b^2 \sin^2 x)}dx$ … (i)

Then, $I = \int_0^{\pi}\frac{(\pi - x)}{(a^2\cos^2(\pi - x) + b^2 \sin^2(\pi - x)]}dx$

or $I = \int_0^{\pi}\frac{(\pi - x)}{(a^2\cos^2 x + b^2 \sin^2 x)}dx.$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi}\frac{(x + \pi - x)}{(a^2\cos^2 x + b^2 \sin^2 x)}dx = \pi\int_0^{\pi}\frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)}$$

$$= 2\pi\int_0^{\pi/2}\frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)} = 2\pi\int_0^{\pi/2}\frac{\sec^2 x}{(a^2 + b^2\tan^2 x)}dx$$

[dividing num. and denom. by $\cos^2 x$]

$$= 2\pi\int_0^{\infty} \frac{dt}{(a^2 + b^2t^2)}, \text{ where } \tan x = t$$

$$= \frac{2\pi}{b^2}\int_0^{\infty} \frac{dt}{\left(\frac{a^2}{b^2} + t^2\right)} = \left[\frac{2\pi}{b^2} \cdot \frac{b}{a}\tan^{-1}\left(\frac{bt}{a}\right)\right]_0^{\infty}$$

$$= \frac{2\pi}{ab}[\tan^{-1}(\infty) - \tan^{-1}(0)] = \frac{2\pi}{ab}\left(\frac{\pi}{2} - 0\right) = \left(\frac{2\pi}{ab} \times \frac{\pi}{2}\right) = \frac{\pi^2}{ab}.$$

$$\therefore \quad I = \frac{\pi^2}{2ab} \Rightarrow \int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)}dx = \frac{\pi^2}{2ab}.$$

EXAMPLE 15 *Prove that* $\int_0^{\pi} x\sin^3 x\,dx = \frac{2\pi}{3}$.

SOLUTION Let $I = \int_0^{\pi} x\sin^3 x\,dx$ … (i)

Then, $I = \int_0^{\pi} (\pi - x)\sin^3(\pi - x)\,dx$

or $I = \int_0^{\pi} (\pi - x)\sin^3 x\,dx$ … (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \pi\sin^3 x\,dx = \pi\int_0^{\pi} \sin^2 x \cdot \sin x\,dx$$

$$= \pi\int_0^{\pi} (1 - \cos^2 x)\sin x\,dx$$

$$= -\pi\int_1^{-1} (1 - t^2)\,dt, \text{ where } \cos x = t$$

$$[x = 0 \Rightarrow t = 1 \text{ and } x = \pi \Rightarrow t = -1]$$

$$= \pi\int_{-1}^{1} (1 - t^2)dt = \pi\left[t - \frac{t^3}{3}\right]_{-1}^{1} = \frac{4\pi}{3}.$$

Hence, $I = \frac{2\pi}{3} \Rightarrow \int_0^{\pi} x\sin^3 x\,dx = \frac{2\pi}{3}$.

EXAMPLE 16 *Evaluate* $\int_0^1 \cot^{-1}\{1 - x + x^2\}\,dx$. **[CBSE 2008]**

SOLUTION We have

$$I = \int_0^1 \cot^{-1}\{1 - x + x^2\}\,dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = \int_0^1 \tan^{-1}\left\{\frac{x+(1-x)}{1-x+x^2}\right\} dx$$

$$= \int_0^1 \{\tan^{-1}x + \tan^{-1}(1-x)\}\, dx$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}\{1-(1-x)\}\, dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \tan^{-1}x\, dx + \int_0^1 \tan^{-1}x\, dx = 2\int_0^1 \tan^{-1}x\, dx$$

$$= 2\int_0^1 (\tan^{-1}x \cdot 1)\, dx$$

$$= 2\left[(\tan^{-1}x)x - \int_0^1 \frac{1}{(1+x^2)} \cdot x\, dx\right]_0^1$$

$$= 2[(\tan^{-1}x)\cdot x]_0^1 - 2\int_0^1 \frac{x}{(1+x^2)} dx$$

$$= 2\{(\tan^{-1}1)\cdot 1 - 0\} - [\log(1+x^2)]_0^1$$

$$= \left(2\times\frac{\pi}{4}\right) - (\log 2 - \log 1) = \left(\frac{\pi}{2} - \log 2\right).$$

EXAMPLE 17 *Evaluate* $\int_{\pi/5}^{3\pi/10} \frac{\sin x}{(\sin x + \cos x)} dx.$ **[CBSE 2009]**

SOLUTION Let $I = \int_{\pi/5}^{3\pi/10} \frac{\sin x}{(\sin x + \cos x)} dx$ … (i)

Here, $a = \frac{\pi}{5}$ and $b = \frac{3\pi}{10} \Rightarrow (a+b) = \left(\frac{\pi}{5} + \frac{3\pi}{10}\right) = \frac{\pi}{2}$.

Using $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$, we have

$$I = \int_{\pi/5}^{3\pi/10} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow \quad I = \int_{\pi/5}^{3\pi/10} \frac{\cos x}{(\sin x + \cos x)} dx \quad \text{… (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_{\pi/5}^{3\pi/10} dx = [x]_{\pi/5}^{3\pi/10} = \left(\frac{3\pi}{10} - \frac{\pi}{5}\right) = \frac{\pi}{10}$$

$$\Rightarrow \quad I = \frac{\pi}{20}.$$

Integrals of the form $\int_{-a}^{a} f(x)\,dx$, *where* $f(x)$ *is even or odd*

We know that

(i) $f(x)$ is odd, if $f(-x) = -f(x)$;

(ii) $f(x)$ is even, if $f(-x) = f(x)$;

(iii) $\int_{-a}^{a} f(x)\,dx = 0$, when $f(x)$ is odd;

(iv) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, when $f(x)$ is even.

EXAMPLE 18 *Show that* $\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0.$

SOLUTION Let $f(x) = \sin^7 x$. Then,

$f(-x) = [\sin(-x)]^7 = -\sin^7 x = -f(x).$

$\therefore$ $f(x)$ is an odd function of x.

But, $\int_{-a}^{a} f(x)\,dx = 0$, when $f(x)$ is odd.

$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0.$

EXAMPLE 19 *Evaluate* $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx.$

SOLUTION Let $f(x) = \sin^2 x.$

Then, $f(-x) = [\sin(-x)]^2 = (-\sin x)^2 = \sin^2 x = f(x).$

$\therefore$ $f(x)$ is an even function.

So, $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = 2\int_{0}^{\pi/2} \sin^2 x\,dx = 2\int_{0}^{\pi/2} \left(\frac{1-\cos 2x}{2}\right) dx$

$$= \int_{0}^{\pi/2} (1 - \cos 2x)\,dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{2}.$$

EXAMPLE 20 *Prove that* $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx = 0.$ **[CBSE 2005C]**

SOLUTION Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$.

Then, $f(-x) = \log\left(\frac{2+x}{2-x}\right) = \log\left(\frac{2-x}{2+x}\right)^{-1} = -\log\left(\frac{2-x}{2+x}\right) = -f(x)$.

$\therefore$ $f(x)$ is an odd function of x.

But, we know that $\int_{-a}^{a} f(x)\,dx = 0$, when $f(x)$ is an odd function of x.

$\therefore$ $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx = 0.$

EXAMPLE 21 *Evaluate* $\int_1^4 f(x)\,dx$, *where* $f(x) = \begin{cases} 4x+3, & \text{if } 1 \le x \le 2 \\ 3x+5, & \text{if } 2 \le x \le 4. \end{cases}$

SOLUTION $\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$

$= \int_1^2 (4x+3)\,dx + \int_2^4 (3x+5)\,dx$

$= [2x^2 + 3x]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^4 = (9+28) = 37.$

INTEGRALS OF MODULUS FUNCTIONS

EXAMPLE 22 *Evaluate:*

(i) $\int_{-1}^{2} |x|\,dx$ *(ii)* $\int_0^1 |5x-3|\,dx$ *(iii)* $\int_0^{\pi} |\cos x|\,dx$

SOLUTION (i) Clearly, $|x| = \begin{cases} -x & \text{when } -1 \le x \le 0 \\ x & \text{when } 0 \le x \le 2. \end{cases}$

$\therefore$ $\int_{-1}^{2} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_0^2 |x|\,dx$

$= \int_{-1}^{0} (-x)\,dx + \int_0^2 x\,dx = \left[\frac{-x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^2 = \left(\frac{1}{2}+2\right) = \frac{5}{2}.$

(ii) Clearly, $|5x-3| = \begin{cases} -(5x-3) & \text{when } 0 \le x \le \frac{3}{5} \\ (5x-3) & \text{when } \frac{3}{5} \le x \le 1. \end{cases}$

$\therefore$ $\int_0^1 |5x-3|\,dx = \int_0^{3/5} |5x-3|\,dx + \int_{3/5}^1 |5x-3|\,dx$

$= \int_0^{3/5} -(5x-3)\,dx + \int_{3/5}^1 (5x-3)\,dx$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^{1}$$

$$= \left(\frac{9}{5} - \frac{9}{10}\right) + \left(\frac{-1}{2} + \frac{9}{10}\right) = \frac{13}{10}.$$

(iii) Clearly, $|\cos x| = \begin{cases} \cos x & \text{when } 0 \le x \le \frac{\pi}{2} \\ -\cos x & \text{when } \frac{\pi}{2} \le x \le \pi. \end{cases}$

$$\therefore \int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = (1+1) = 2.$$

EXAMPLE 23 *Evaluate* $\int_1^4 f(x)\,dx$, *where* $f(x) = |x-1| + |x-2| + |x-3|$. **[CBSE 2013]**

SOLUTION $\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx$

$$= \int_1^2 \{(x-1) - (x-2) - (x-3)\}\,dx + \int_2^3 \{(x-1) + (x-2) - (x-3)\}\,dx$$

$$+ \int_3^4 \{(x-1) + (x-2) + (x-3)\}\,dx$$

$$= \int_1^2 (-x+4)\,dx + \int_2^3 x\,dx + \int_3^4 (3x-6)\,dx$$

$$= \left[\frac{-x^2}{2} + 4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2} - 6x\right]_3^4 = \left(\frac{5}{2} + \frac{5}{2} + \frac{9}{2}\right) = \frac{19}{2}.$$

EXAMPLE 24 *Evaluate:*

(i) $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx$ *(ii)* $\int_{-1}^{1} e^{|x|}\,dx$ *(iii)* $\int_{-2}^{1} |2x+1|\,dx$

SOLUTION (i) Clearly, $|\sin x|$ is an even function of x.

$$\therefore \int_{-\pi/2}^{\pi/2} |\sin x|\,dx = 2\int_0^{\pi/2} |\sin x|\,dx$$

$$= 2\int_0^{\pi/2} \sin x\,dx \quad \left[\because \sin x \ge 0, \text{ when } 0 \le x \le \frac{\pi}{2}\right]$$

$$= [-2\cos x]_0^{\pi/2} = 2.$$

(ii) Clearly, $e^{|x|}$ is an even function of x.

$$\therefore \int_{-1}^{1} e^{|x|}\,dx = 2\int_{0}^{1} e^{|x|}\,dx$$

$$= 2\int_{0}^{1} e^{x}\,dx \qquad [\because\ |x| = x,\ \text{when}\ \ 0 \le x \le 1]$$

$$= [2e^{x}]_{0}^{1} = (2e - 2) = 2(e - 1).$$

(iii) $\left[-2 \le x < -\frac{1}{2} \Rightarrow 2x + 1 < 0\right]$ and $\left[-\frac{1}{2} \le x \le 1 \Rightarrow 2x + 1 \ge 0\right]$

$$\therefore \int_{-2}^{1} |2x+1|\,dx = \int_{-2}^{-1/2} |2x+1|\,dx + \int_{-1/2}^{1} |2x+1|\,dx$$

$$= \int_{-2}^{-1/2} -(2x+1)\,dx + \int_{-1/2}^{1} (2x+1)\,dx$$

$$= [-x^2 - x]_{-2}^{-1/2} + [x^2 + x]_{-1/2}^{1}$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (-4 + 2) + \left[2 - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} + 2 + \frac{9}{4} = \frac{9}{2}.$$

EXAMPLE 25 *Evaluate* $\int_{0}^{2\pi} |\sin x|\,dx.$

SOLUTION We know that $\sin x$ is positive, when $0 \le x \le \pi$ and $\sin x$ is negative when $\pi \le x \le 2\pi$.

$$\therefore \quad |\sin x| = \begin{cases} \sin x, & \text{when}\ \ 0 \le x \le \pi \\ -\sin x, & \text{when}\ \ \pi \le x \le 2\pi. \end{cases}$$

$$\therefore \int_{0}^{2\pi} |\sin x|\,dx = \int_{0}^{\pi} |\sin x|\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx$$

$$= \int_{0}^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx$$

$$= [-\cos x]_{0}^{\pi} + [\cos x]_{\pi}^{2\pi} = (2 + 2) = 4.$$

EXAMPLE 26 *Evaluate* $\int_{-\pi/2}^{\pi/2} f(x)\,dx$, *where* $f(x) = \sin|x| + \cos|x|$. **[CBSE 2000]**

SOLUTION $f(x) = \sin|x| + \cos|x|$

$\Rightarrow\ f(-x) = \sin|-x| + \cos|-x| = \sin|x| + \cos|x| = f(x)$

$\Rightarrow\ f(x)$ is an even function.

$$\therefore\ I = \int_{-\pi/2}^{\pi/2} f(x)\,dx$$

$$= 2\int_0^{\pi/2} f(x)\,dx = 2\int_0^{\pi/2} \{\sin|x| + \cos|x|\}\,dx$$

$$= 2\int_0^{\pi/2} (\sin x + \cos x)\,dx \qquad [\because\ |x| = x \text{ in } 0 < x < \frac{\pi}{2}]$$

$$= 2\cdot[-\cos x + \sin x]_0^{\pi/2} = 2\left[\left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-\cos 0 + \sin 0)\right]$$

$$= 4.$$

EXERCISE 16C

Prove that

1. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}dx = \frac{\pi}{4}$

2. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})}dx = \frac{\pi}{4}$

3. (i) $\displaystyle\int_0^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)}dx = \frac{\pi}{4}$ **[CBSE 1998, 2000C, '04]**

   (ii) $\displaystyle\int_0^{\pi/2} \frac{\cos^3 x\,dx}{(\sin^3 x + \cos^3 x)} = \frac{\pi}{4}$ **[CBSE 2004C]**

4. (i) $\displaystyle\int_0^{\pi/2} \frac{\sin^7 x}{(\sin^7 x + \cos^7 x)}dx = \frac{\pi}{4}$ **[CBSE 2000C]**

   (ii) $\displaystyle\int_0^{\pi/2} \frac{\sin^5 x}{(\sin^5 x + \cos^5 x)}dx = \frac{\pi}{4}$ **[CBSE 2000C]**

5. $\displaystyle\int_0^{\pi/2} \frac{\cos^4 x}{(\sin^4 x + \cos^4 x)}dx = \frac{\pi}{4}$

6. $\displaystyle\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x + \cos^{1/4} x)}dx = \frac{\pi}{4}$

7. $\displaystyle\int_0^{\pi/2} \frac{\sin^{3/2} x}{(\sin^{3/2} x + \cos^{3/2} x)}dx = \frac{\pi}{4}$

8. $\displaystyle\int_0^{\pi/2} \frac{\sin^n x}{(\sin^n x + \cos^n x)}dx = \frac{\pi}{4}$

9. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\sqrt{\tan x} + \sqrt{\cot x})}dx = \frac{\pi}{4}$

10. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(\sqrt{\tan x} + \sqrt{\cot x})}dx = \frac{\pi}{4}$

11. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \tan x)} = \frac{\pi}{4}$ **[CBSE 2003C, '06C]**

12. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \cot x)} = \frac{\pi}{4}$

13. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \tan^3 x)} = \frac{\pi}{4}$

14. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \cot^3 x)} = \frac{\pi}{4}$

15. $\displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \sqrt{\tan x})} = \frac{\pi}{4}$ **[CBSE 2006C, '07C, '11]**

16. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(1 + \sqrt{\cot x})}dx = \frac{\pi}{4}$

17. $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(1+\sqrt{\tan x})} dx = \frac{\pi}{4}$

18. $\int_0^{\pi/2} \frac{(\sin x - \cos x)}{(1+\sin x \cos x)} dx = 0$

19. $\int_0^1 x(1-x)^5 dx = \frac{1}{42}$

20. $\int_0^2 x\sqrt{2-x}\, dx = \frac{16\sqrt{2}}{15}$ **[CBSE 2005C]**

21. $\int_0^{\pi} x\cos^2 x\, dx = \frac{\pi^2}{4}$

22. $\int_0^{\pi} \frac{x\tan x}{(\sec x \operatorname{cosec} x)} dx = \frac{\pi^2}{4}$ **[CBSE 2008, '09C, '11C]**

23. $\int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$

24. $\int_0^{\pi} \frac{x\tan x}{(\sec x + \cos x)} dx = \frac{\pi^2}{4}$

25. $\int_0^{\pi} \frac{x\sin x}{(1+\sin x)} dx = \pi\left(\frac{\pi}{2} - 1\right)$

26. $\int_0^{\pi} \frac{x}{(1+\sin^2 x)} dx = \frac{\pi^2}{2\sqrt{2}}$

27. $\int_0^{\pi/2} (2\log\cos x - \log\sin 2x)\, dx = -\frac{\pi}{2}(\log 2)$

28. $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx = \frac{\pi}{4}$

29. $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$

30. $\int_0^a \frac{\sqrt{x}}{(\sqrt{x}+\sqrt{a-x})} dx = \frac{\pi}{4}$ **[CBSE 2008C]**

31. $\int_0^{\pi} \sin^2 x \cos^3 x\, dx = 0$

32. $\int_0^{\pi} \sin^{2m} x \cos^{2m+1} x\, dx = 0$, where m is a positive integer

33. $\int_0^{\pi/2} (\sin x - \cos x)\log(\sin x + \cos x)\, dx = 0$

34. $\int_0^{\pi/2} \log(\sin 2x)\, dx = -\frac{\pi}{2}(\log 2)$

35. $\int_0^{\pi} x\log(\sin x)\, dx = -\frac{\pi^2}{2}(\log 2)$

36. $\int_0^{\pi} \log(1+\cos x)\, dx = -\pi(\log 2)$

37. $\int_0^{\pi/2} \log(\tan x + \cot x)\, dx = \pi(\log 2)$

38. $\int_{\pi/8}^{3\pi/8} \frac{\cos x}{(\cos x + \sin x)} dx = \frac{\pi}{8}$

39. $\int_{\pi/6}^{\pi/3} \frac{1}{(1+\sqrt{\tan x})} dx = \frac{\pi}{12}$ **[CBSE 2005, '06C, '07, '11, '12C]**

40. $\int_{\pi/4}^{3\pi/4} \frac{dx}{(1+\cos x)} = 2$

41. $\int_{\pi/4}^{3\pi/4} \frac{x}{(1+\sin x)} dx = \pi(\sqrt{2}-1)$

42. $\int_{\alpha/4}^{3\alpha/4} \frac{\sqrt{x}}{(\sqrt{a-x}+\sqrt{x})} dx = \frac{a}{4}$

43. $\int_{1}^{4} \frac{\sqrt{x}}{(\sqrt{5-x}+\sqrt{x})} dx = \frac{3}{2}$

44. $\int_{0}^{\pi/2} x \cot x \, dx = \frac{\pi}{2}(\log 2)$

45. $\int_{0}^{1} \left(\frac{\sin^{-1} x}{x}\right) dx = \frac{\pi}{2}(\log 2)$

46. $\int_{0}^{1} \frac{\log x}{\sqrt{1-x^2}} dx = -\frac{\pi}{2}(\log 2)$

47. $\int_{0}^{1} \frac{\log (1+x)}{(1+x^2)} dx = \frac{\pi}{8}(\log 2)$

[CBSE 2011C]

48. $\int_{-a}^{a} x^3 \sqrt{a^2 - x^2} dx = 0$

49. $\int_{-\pi}^{\pi} (\sin^{75} x + x^{125}) \, dx = 0$

50. $\int_{-\pi}^{\pi} x^{12} \sin^9 x \, dx = 0$

51. $\int_{-1}^{1} e^{|x|} dx = 2(e-1)$

52. $\int_{-2}^{2} |x+1| dx = 6$

53. $\int_{0}^{8} |x-5| dx = 17$

54. $\int_{0}^{2\pi} |\cos x| dx = 4.$

55. $\int_{-\pi/4}^{\pi/4} |\sin x| dx = (2-\sqrt{2})$

56. Let $f(x) = \begin{cases} 2x+1, & \text{when } 1 \le x \le 2 \\ x^2+1, & \text{when } 2 \le x \le 3. \end{cases}$

Show that $\int_{1}^{3} f(x) dx = \frac{34}{3}$.

57. Let $f(x) = \begin{cases} 3x^2+4, & \text{when } 0 \le x \le 2 \\ 9x-2, & \text{when } 2 \le x \le 4. \end{cases}$

Show that $\int_{0}^{4} f(x) dx = 66.$

58. Prove that $\int_{0}^{4} \{|x| + |x-2| + |x-4| \, dx\} = 20$ **[CBSE 2013]**

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16C)

11. $I = \int_{0}^{\pi/2} \frac{dx}{\left(1+\frac{\sin x}{\cos x}\right)} = \int_{0}^{\pi/2} \frac{\cos x}{(\sin x + \cos x)} dx.$

14. $I = \int_{0}^{\pi/2} \frac{dx}{\left(1+\frac{\cos^3 x}{\sin^3 x}\right)} = \int_{0}^{\pi/2} \frac{\sin^3 x}{(\sin^3 x + \cos^3 x)} dx.$

17. $I=\int_0^{\pi/2}\frac{\sqrt{\frac{\sin x}{\cos x}}}{\left(1+\sqrt{\frac{\sin x}{\cos x}}\right)}dx=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{(\sqrt{\sin x}+\sqrt{\cos x})}dx.$

19. $I=\int_0^1(1-x)[1-(1-x)]^5\,dx=\int_0^1(1-x)x^5dx=\int_0^1(x^5-x^6)\,dx.$

20. $I=\int_0^2(2-x)[2-(2-x)]^{1/2}\,dx=\int_0^2(2-x)\sqrt{x}\,dx=\int_0^2(2\sqrt{x}-x^{3/2})\,dx.$

21. $I=\int_0^{\pi}x\cos^2x\,dx$ and $I=\int_0^{\pi}(\pi-x)\cos^2(\pi-x)\,dx=\int_0^{\pi}(\pi-x)\cos^2x\,dx.$

$\therefore\quad 2I=\pi\int_0^{\pi}\cos^2x\,dx=\frac{\pi}{2}\cdot\int_0^{\pi}(1+\cos 2x)\,dx\ \Rightarrow\ I=\frac{\pi^2}{4}\cdot$

22. $I=\int_0^{\pi}x\sin^2x\,dx$ and $I=\int_0^{\pi}(\pi-x)\sin^2(\pi-x)\,dx=\int_0^{\pi}(\pi-x)\sin^2x\,dx.$

$\therefore\quad 2I=\pi\cdot\int_0^{\pi}\sin^2x\,dx=\frac{\pi}{2}\cdot\int_0^{\pi}(1-\cos 2x)\,dx.$

24. $I=\int_0^{\pi}\frac{x\sin x}{(1+\cos^2x)}dx$ and $I=\int_0^{\pi}\frac{(\pi-x)\sin x}{(1+\cos^2x)}dx.$

$\therefore\quad 2I=\pi\cdot\int_0^{\pi}\frac{\sin x}{(1+\cos^2x)}dx.$ Now, put $\cos x=t.$

25. $2I=\pi\cdot\int_0^{\pi}\frac{\sin x}{(1+\sin x)}dx=\pi\cdot\int_0^{\pi}20\left(1-\frac{1}{1+\sin x}\right)dx$

$=\pi^2-\pi\cdot\int_0^{\pi}\frac{(1-\sin x)}{(1-\sin^2x)}dx=\pi^2-\pi\cdot\int_0^{\pi}\frac{(1-\sin x)}{\cos^2x}dx$

$=\pi^2-\pi\cdot\int_0^{\pi}(\sec^2x-\sec x\tan x)\,dx.$

26. $2I=2\pi\cdot\int_0^{\pi/2}\frac{dx}{(1+\sin^2x)}\cdot$

Divide num. and denom. by $\cos^2x$ and put $\tan x=t.$

28. Putting $x=\tan\theta$, we get $I=\int_0^{\pi/2}\frac{\sin\theta}{(\cos\theta+\sin\theta)}d\theta.$

29. Put $x=a\sin\theta$ and $dx=a\cos\theta\,d\theta.$

30. Apply $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx.$

In Q. 31 to Q. 33, apply $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ to get $I=-I.$

34. Putting $2x = t$, we get $I = \frac{1}{2} \cdot \int_0^{\pi} \log(\sin t)\, dt = \int_0^{\pi/2} \log(\sin t)\, dt$. [by Thm. 8]

35. $2I = \pi \cdot \int_0^{\pi} \log(\sin x)\, dx = 2\pi \cdot \int_0^{\pi/2} \log(\sin x)\, dx$. [by Thm. 8]

36. $2I = \int_0^{\pi} \log(1 - \cos^2 x)\, dx = 2\int_0^{\pi} \log(\sin x)\, dx = 4\int_0^{\pi/2} \log(\sin x)\, dx$. [Thm. 8]

37. $I = \int_0^{\pi/2} \log\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) dx = \int_0^{\pi/2} \log\left(\frac{1}{\sin x \cos x}\right) dx$

$= -\left[\int_0^{\pi/2} \log(\sin x)\, dx + \int_0^{\pi/2} \log(\cos x)\, dx\right]$.

In Q. 38 to Q. 43, apply, $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$.

44. Integrating by parts, we get

$I = [x \log(\sin x)]_0^{\pi/2} - \int_0^{\pi/2} \log(\sin x)\, dx = -\int_0^{\pi/2} \log(\sin x)\, dx$.

45. Put $x = \sin\theta$ and integrate by parts.

46. Put $x = \sin\theta$.

47. Put $x = \tan\theta$.

In Q.48 to Q.50, $f(-x) = -f(x) \Rightarrow I = 0$.

51. $I = \int_{-1}^{0} e^{-x} dx + \int_0^1 e^x dx$.

58. $I = \int_0^2 \{|x| + |x-2| + |x-4|\} dx + \int_2^4 |x| + |x-2| + |x-4|\} dx$

$= \int_0^2 \{x - (x-2) - (x-4)\} dx + \int_2^4 \{x + (x-2) - (x-4)\} dx$

$= \int_0^2 (x - x + 2 - x + 4)\, dx + \int_2^4 (x + x - 2 - x + 4)\, dx$

$= \int_0^2 (-x + 2)\, dx + \int_2^4 (x + 2)\, dx$.

Definite Integral as the Limit of a Sum

Let $f(x)$ be a continuous real-valued function, defined in the closed interval $[a, b]$. Then we define

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f[a + (n-1)h)],$$

where $nh = (b - a)$.

The method of evaluating $\int_a^b f(x)\,dx$ by using the above definition is called *integration from first principles*.

Some Useful Results for Direct Applications

(i) $1+2+3+\ldots+(n-1)=\frac{1}{2}n(n-1).$

(ii) $1^2+2^2+3^2+\ldots+(n-1)^2=\frac{1}{6}(n-1)n(2n-1).$

(iii) $1^3+2^3+3^3+\ldots+(n-1)^3=\left\{\frac{n(n-1)}{2}\right\}^2.$

(iv) $a+ar+ar^2+\ldots+ar^{n-1}=\frac{a(r^n-1)}{(r-1)}.$

(v) $\sin a+\sin(a+h)+\sin(a+2h)+\ldots+\sin[a+(n-1)h]$

$$=\frac{\sin\left\{a+\left(\frac{n-1}{2}\right)h\right\}\sin\left(\frac{nh}{2}\right)}{\sin(h/2)}.$$

(vi) $\cos a+\cos(a+h)+\cos(a+2h)+\ldots+\cos[a+(n-1)h]$

$$=\frac{\cos\left\{a+\frac{(n-1)}{2}h\right\}\sin\left(\frac{nh}{2}\right)}{\sin(h/2)}.$$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^5 (x+1)\,dx$ *(ii)* $\int_1^3 (2x+3)\,dx$ **[CBSE 2000C]**

SOLUTION (i) Let $f(x)=(x+1);\ a=0;\ b=5$ and $nh=(5-0)=5$. Then,

$$\int_0^5 (x+1)\,dx=\lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\ldots+f\{0+(n-1)h\}]$$

$$=\lim_{h\to 0} h[1+(h+1)+(2h+1)+\ldots+\{(n-1)h+1\}]$$

$$=\lim_{h\to 0} h[n+\{h+2h+3h+\ldots+(n-1)h\}]$$

$$=\lim_{h\to 0} h[n+\{1+2+3+\ldots+(n-1)\}h]$$

$$=\lim_{h\to 0} h\left[n+\frac{n(n-1)}{2}h\right]=\lim_{h\to 0}\left[nh+\frac{nh(nh-h)}{2}\right]$$

$$= \lim_{h\to 0}\left[5 + \frac{5(5-h)}{2}\right] \qquad [\because\ nh = 5]$$

$$= \frac{35}{2}.$$

(ii) Let $f(x) = (2x + 3);\ a = 1;\ b = 3$ and $nh = (3 - 1) = 2.$

Then, $\int_1^3 (2x + 3)\,dx = \lim_{h\to 0} h[f(1) + f(1 + h) + \ldots + f\{1 + (n-1)h\}]$

$$= \lim_{h\to 0} h[5 + (5 + 2h) + (5 + 4h) + \ldots + \{5 + 2(n-1)h\}]$$

$$[\because\ f(1) = 5,\ f(1 + h) = 5 + 2h, \text{ etc.}]$$

$$= \lim_{h\to 0} h[5n + \{2h + 4h + \ldots + 2(n-1)h\}]$$

$$= \lim_{h\to 0} h[5n + 2\{1 + 2 + 3 + \ldots + (n-1)\}h]$$

$$= \lim_{h\to 0} h\left[5n + 2\cdot\frac{n(n-1)}{2}h\right]$$

$$= \lim_{h\to 0} [5nh + nh(nh - h)]$$

$$= \lim_{h\to 0} [10 + 2(2 - h)] \qquad [\because\ nh = 2]$$

$$= 14.$$

EXAMPLE 2 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^2 (x^2 + 1)\,dx$ *(ii)* $\int_1^3 (x^2 + x)\,dx$ **[CBSE 2006C]**

SOLUTION (i) Let $f(x) = (x^2 + 1);\ a = 0;\ b = 2$ and $nh = (2 - 0) = 2.$

$$\therefore \int_0^2 (x^2 + 1)\,dx = \lim_{h\to 0} h[f(0) + f(0 + h) + \ldots + f\{0 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[1 + (h^2 + 1) + (4h^2 + 1) + (9h^2 + 1) +$$

$$\ldots + \{(n-1)^2h^2 + 1\}]$$

$$[\because\ f(0) = 1,\ f(0 + h) = (h^2 + 1),\ f(0 + 2h) = (4h^2 + 1), \text{ etc.}]$$

$$= \lim_{h\to 0} h[n + \{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\}h^2]$$

$$= \lim_{h\to 0} h\left[n + \frac{(n-1)n\,(2n-1)}{6}\cdot h^2\right]$$

$$= \lim_{h\to 0}\left[nh + \frac{(nh - h)nh\,(2nh - h)}{6}\right]$$

$$= \lim_{h\to 0}\left[2 + \frac{(2-h)2\,(4-h)}{6}\right] \qquad [\because\ nh = 2]$$

$$= \frac{14}{3}.$$

(ii) Let $f(x) = (x^2 + x)$; $a = 1$; $b = 3$ and $nh = (3 - 1) = 2$.

$$\therefore \int_1^3 (x^2 + x)\,dx = \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots + f\{1 + (n-1)h\}]$$

$$= \lim_{h \to 0} h[(1^2 + 1) + \{(1+h)^2 + (1+h)\} + \{(1+2h)^2 + (1+2h)\} + \ldots + \{1 + (n-1)h\}^2 + \{1 + (n-1)h\}]$$

$$= \lim_{h \to 0} h[2n + h^2\{1^2 + 2^2 + \ldots + (n-1)^2\} + 3h\{1 + 2 + \ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[2n + h^2 \cdot \frac{(n-1)n\,(2n-1)}{6} + 3h \cdot \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0} \left[2nh + \frac{(nh-h)nh\,(2nh-h)}{6} + \frac{3}{2} \cdot nh(nh-h)\right]$$

$$= \lim_{h \to 0} \left[4 + \frac{(2-h)2\,(4-h)}{6} + \frac{3}{2} \cdot 2(2-h)\right] \quad [\because\ nh = 2]$$

$$= \frac{38}{3}.$$

EXAMPLE 3 *Evaluate* $\int_0^1 (3x^2 + 2x + 1)\,dx$ *as limit of sums.* **[CBSE 2007C]**

SOLUTION Let $f(x) = (3x^2 + 2x + 1)$; $a = 0$, $b = 1$ and $nh = (1 - 0) = 1$.

Then, $\int_0^1 (3x^2 + 2x + 1)\,dx$

$$= \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0 + (n-1)h\}]$$

$$= \lim_{h \to 0} h[1 + (3h^2 + 2h + 1) + (3 \cdot 2^2h^2 + 2 \cdot 2h + 1) + \ldots + \{3 \cdot (n-1)^2h^2 + 2 \cdot (n-1)h + 1\}]$$

$$= \lim_{h \to 0} h[n + 3h^2\{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\} + 2h\{1 + 2 + 3 + \ldots + (n-1)\}]$$

$$= \lim_{h \to 0} h\left[n + 3h^2 \cdot \frac{(n-1)n\,(2n-1)}{6} + 2h \cdot \frac{n(n-1)}{2}\right]$$

$$= \lim_{h \to 0} \left[nh + \frac{1}{2}(nh-h)(nh)(2nh-h) + (nh)(nh-h)\right]$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{2}(1-h) \cdot 1 \cdot (2-h) + 1 \cdot (1-h)\right] \quad [\because\ nh = 1]$$

$$= 3.$$

EXAMPLE 4 *Evaluate* $\int_{-1}^1 e^x\,dx$ *as limit of sums.*

SOLUTION Let $f(x) = e^x$ and $nh = \{1 - (-1)\} = 2$. Then,

$$\int_{-1}^{1} e^x dx = \lim_{h\to 0} h[f(-1) + f(-1+h) + \ldots + f\{-1+(n-1)h\}]$$

$$= \lim_{h\to 0} h[e^{-1} + e^{(-1+h)} + e^{(-1+2h)} + \ldots + e^{\{-1+(n-1)h\}}]$$

$$[\because\ f(-1) = e^{-1},\ f(-1+h) = e^{(-1+h)}, \text{ etc.}]$$

$$= \lim_{h\to 0} e^{-1} h[1 + e^h + e^{2h} + \ldots + e^{(n-1)h}]$$

$$= \frac{1}{e} \cdot \lim_{h\to 0} h\left[\frac{(e^h)^n - 1}{(e^h - 1)}\right] = \frac{1}{e} \cdot \lim_{h\to 0} \frac{h[e^{nh} - 1]}{(e^h - 1)}$$

$$= \frac{1}{e} \cdot \lim_{h\to 0} \frac{h(e^2 - 1)}{(e^h - 1)} \qquad [\because\ nh = 2]$$

$$= \frac{1}{e} \cdot \lim_{h\to 0} \frac{h(e^2 - 1)}{\left\{1 + h + \frac{h^2}{\lfloor 2} + \frac{h^3}{\lfloor 3} + \ldots\right\} - 1}$$

$$= \frac{1}{e} \cdot \lim_{h\to 0} \frac{e^2 - 1}{\left(1 + \frac{h}{\lfloor 2} + \frac{h^2}{\lfloor 3} + \ldots\right)} = \frac{1}{e}(e^2 - 1) = \left(e - \frac{1}{e}\right).$$

EXAMPLE 5 *Evaluate* $\int_a^b \sin x\, dx$ *from first principles.*

SOLUTION Let $f(x) = \sin x$ and let $nh = (b-a)$. Then,

$$\int_a^b \sin x\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + \ldots + f\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h[\sin a + \sin(a+h) + \ldots + \sin\{a+(n-1)h\}]$$

$$= \lim_{h\to 0} h \cdot \frac{\sin\left\{a + \left(\frac{n-1}{2}\right)h\right\} \sin\left(\frac{nh}{2}\right)}{\sin(h/2)}$$

$$= \lim_{h\to 0} \left[2\sin\left\{a + \frac{1}{2}nh - \frac{1}{2}h\right\} \sin\left(\frac{nh}{2}\right) \times \frac{(h/2)}{\sin(h/2)}\right]$$

$$= \lim_{h\to 0} \left[2\sin\left\{a + \frac{1}{2}(b-a) - \frac{1}{2}h\right\} \sin\left(\frac{b-a}{2}\right) \times \frac{(h/2)}{\sin(h/2)}\right]$$

$$[\because\ nh = (b-a)]$$

$$= \lim_{h\to 0} 2\sin\left\{\left(\frac{b+a}{2}\right) - \frac{1}{2}h\right\} \sin\left(\frac{b-a}{2}\right) \cdot \lim_{h\to 0} \frac{(h/2)}{\sin(h/2)}$$

$$= 2\sin\left(\frac{b+a}{2}\right) \sin\left(\frac{b-a}{2}\right)$$

$$= (\cos a - \cos b) \quad [\because\ 2\sin A \sin B = \cos(A-B) - \cos(A+B)].$$

EXERCISE 16D

Evaluate each of the following integrals as the limit of sums:

1. $\int_0^2 (x+4)\,dx$ 2. $\int_1^2 (3x-2)\,dx$ 3. $\int_1^3 x^2 dx$

4. $\int_0^3 (x^2+1)\,dx$ **[CBSE 2001]** 5. $\int_2^5 (3x^2-5)\,dx$ **[CBSE 2009C]**

6. $\int_0^3 (x^2+2x)\,dx$ 7. $\int_1^4 (3x^2+2x)\,dx$

8. $\int_1^3 (x^2+5x)\,dx$ **[CBSE 2008C, '10]** 9. $\int_1^3 (2x^2+5x)\,dx$ **[CBSE 2012C]**

10. $\int_0^2 x^3 dx$ 11. $\int_2^4 (x^2-3x+2)\,dx$

12. $\int_0^2 (x^2+x)\,dx$ **[CBSE 2005]** 13. $\int_0^3 (2x^2+3x+5)\,dx$ **[CBSE 2007]**

14. $\int_0^1 |3x-1|\,dx$ 15. $\int_0^2 e^x dx$

16. $\int_1^3 e^{-x} dx$ 17. $\int_a^b \cos x\,dx$

ANSWERS (EXERCISE 16D)

1. 10 **2.** $\frac{5}{2}$ **3.** $\frac{26}{3}$ **4.** 12 **5.** 102 **6.** 18 **7.** 78 **8.** $\frac{86}{3}$

9. $\frac{112}{3}$ **10.** 4 **11.** $\frac{14}{3}$ **12.** $\frac{14}{3}$ **13.** $\frac{93}{2}$ **14.** 1 **15.** (e^2-1)

16. $\frac{(e^2-1)}{e^3}$ **17.** $(\sin b - \sin a)$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 16D)

14. $\int_0^1 |3x-1| = \int_0^{1/3} (1-3x)\,dx + \int_{1/3}^1 (3x-1)\,dx = I_1 + I_2.$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $\int_1^4 x\sqrt{x}\,dx = ?$

(a) 12.8 (b) 12.4 (c) 7 (d) none of these

2. $\int_0^2 \sqrt{6x+4}\,dx = ?$

(a) $\frac{64}{9}$ (b) 7 (c) $\frac{56}{9}$ (d) $\frac{60}{9}$

3. $\int_0^1 \frac{dx}{\sqrt{5x+3}} = ?$

(a) $\frac{2}{5}(\sqrt{8}-\sqrt{3})$ (b) $\frac{2}{5}(\sqrt{8}+\sqrt{3})$ (c) $\frac{2}{5}\sqrt{8}$ (d) none of these

4. $\int_0^1 \frac{1}{(1+x^2)}dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) none of these

5. $\int_0^2 \frac{dx}{\sqrt{4-x^2}} = ?$

(a) 1 (b) $\sin^{-1}\frac{1}{2}$ (c) $\frac{\pi}{4}$ (d) none of these

6. $\int_{\sqrt{3}}^{\sqrt{8}} x\sqrt{1+x^2}\,dx = ?$

(a) $\frac{19}{3}$ (b) $\frac{19}{6}$ (c) $\frac{38}{3}$ (d) $\frac{9}{4}$

7. $\int_0^1 \frac{x^3}{(1+x^8)}dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{16}$

8. $\int_1^e \frac{(\log x)^2}{x}dx = ?$

(a) $\frac{1}{3}$ (b) $\frac{1}{3}e^3$ (c) $\frac{1}{3}(e^3-1)$ (d) none of these

9. $\int_{\pi/4}^{\pi/2} \cot x\,dx = ?$

(a) $\log 2$ (b) $2\log 2$ (c) $\frac{1}{2}\log 2$ (d) none of these

10. $\int_{0}^{\pi/4} \tan^2 x\,dx = ?$

(a) $\left(1-\frac{\pi}{4}\right)$ (b) $\left(1+\frac{\pi}{4}\right)$ (c) $\left(1-\frac{\pi}{2}\right)$ (d) $\left(1+\frac{\pi}{2}\right)$

11. $\int_{0}^{\pi/2} \cos^2 x\,dx = ?$

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{4}$ (d) 1

12. $\int_{\pi/3}^{\pi/2} \text{cosec}\, x\,dx = ?$

(a) $\frac{1}{2}\log 2$ (b) $\frac{1}{2}\log 3$ (c) $-\log 2$ (d) none of these

13. $\int_{0}^{\pi/2} \cos^3 x\,dx = ?$

(a) 1 (b) $\frac{3}{4}$ (c) $\frac{2}{3}$ (d) none of these

14. $\int_{0}^{\pi/4} \frac{e^{\tan x}}{\cos^2 x}dx = ?$

(a) $(e-1)$ (b) $(e+1)$ (c) $\left(\frac{1}{e}+1\right)$ (d) $\left(\frac{1}{e}-1\right)$

15. $\int_{0}^{\pi/2} \frac{\cos x}{(1+\sin^2 x)}dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) none of these

16. $\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2}dx = ?$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) none of these

17. $\int_{0}^{\pi} \frac{dx}{(1+\sin x)} = ?$

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 0

18. $\int_0^{\pi/2} (\sqrt{\sin x}\cos x)^3 dx = ?$

(a) $\frac{2}{9}$ (b) $\frac{2}{15}$ (c) $\frac{8}{45}$ (d) $\frac{5}{2}$

19. $\int_0^1 \frac{xe^x}{(1+x)^2} dx = ?$

(a) $\left(\frac{e}{2}-1\right)$ (b) $(e-1)$ (c) $e(e-1)$ (d) none of these

20. $\int_0^{\pi/2} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx = ?$

(a) 0 (b) $\frac{\pi}{4}$ (c) $e^{\pi/2}$ (d) $(e^{\pi/2}-1)$

21. $\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx = ?$

(a) 0 (b) 1 (c) 2 (d) $\sqrt{2}$

22. $\int_0^{\pi/2} \sqrt{1+\cos 2x}\, dx = ?$

(a) $\sqrt{2}$ (b) $\frac{3}{2}$ (c) $\sqrt{3}$ (d) 2

23. $\int_0^1 \frac{(1-x)}{(1+x)} dx = ?$

(a) $\frac{1}{2}\log 2$ (b) $(2\log 2+1)$ (c) $(2\log 2-1)$ (d) $\left(\frac{1}{2}\log 2-1\right)$

24. $\int_0^{\pi/2} \sin^2 x\, dx = ?$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

25. $\int_0^{\pi/6} \cos x \cos 2x\, dx = ?$

(a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{3}$ (d) $\frac{7}{12}$

26. $\int_0^{\pi/2} \sin x \sin 2x\, dx = ?$

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{5}{6}$ (d) $\frac{3}{5}$

27. $\int_0^{\pi} (\sin 2x \cos 3x)\, dx = ?$

(a) $\frac{4}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{5}{12}$ (d) $-\frac{12}{5}$

28. $\int_0^1 \frac{dx}{(e^x + e^{-x})} = ?$

(a) $\left(1 - \frac{\pi}{4}\right)$ (b) $\tan^{-1} e$ (c) $\tan^{-1} e + \frac{\pi}{4}$ (d) $\tan^{-1} e - \frac{\pi}{4}$

29. $\int_0^9 \frac{dx}{(1+\sqrt{x})} = ?$

(a) $(3 - 2\log 2)$ (b) $(3 + 2\log 2)$ (c) $(6 - 2\log 4)$ (d) $(6 + 2\log 4)$

30. $\int_0^{\pi/2} x \cos x\, dx = ?$

(a) $\frac{\pi}{2}$ (b) $\left(\frac{\pi}{2} - 1\right)$ (c) $\left(\frac{\pi}{2} + 1\right)$ (d) none of these

31. $\int_0^1 \frac{dx}{(1 + x + x^2)} = ?$

(a) $\frac{\pi}{\sqrt{3}}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{3\sqrt{3}}$ (d) none of these

32. $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\left(\frac{\pi}{2} - 1\right)$ (c) $\left(\frac{\pi}{2} + 1\right)$ (d) none of these

33. $\int_0^1 \frac{(1-x)}{(1+x)} dx = ?$

(a) $(\log 2 + 1)$ (b) $(\log 2 - 1)$ (c) $(2\log 2 - 1)$ (d) $(2\log 2 + 1)$

34. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx = ?$

(a) $a\pi$ (b) $\frac{a\pi}{2}$ (c) $2a\pi$ (d) none of these

35. $\int_0^{\sqrt{2}} \sqrt{2 - x^2}\, dx = ?$

(a) π (b) 2π (c) $\frac{\pi}{2}$ (d) none of these

36. $\int_{-2}^{2} |x|\,dx = ?$

(a) 4 (b) 3.5 (c) 2 (d) 0

37. $\int_{0}^{1} |2x-1|\,dx = ?$

(a) 2 (b) $\frac{1}{2}$ (c) 1 (d) 0

38. $\int_{-2}^{1} |2x+1|\,dx = ?$

(a) $\frac{5}{2}$ (b) $\frac{7}{2}$ (c) $\frac{9}{2}$ (d) 4

39. $\int_{-2}^{1} \frac{|x|}{x}\,dx = ?$

(a) 3 (b) 2.5 (c) 1.5 (d) none of these

40. $\int_{-a}^{a} x|x|\,dx = ?$

(a) 0 (b) $2a$ (c) $\frac{2a^3}{3}$ (d) none of these

41. $\int_{0}^{\pi} |\cos x|\,dx = ?$

(a) 2 (b) $\frac{3}{2}$ (c) 1 (d) 0

42. $\int_{0}^{2\pi} |\sin x|\,dx = ?$

(a) 2 (b) 4 (c) 1 (d) none of these

43. $\int_{0}^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx = ?$

(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{4}$

44. $\int_{0}^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\cos x} + \sqrt{\sin x})}\,dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) π (d) 0

45. $\int_{0}^{\pi/2} \frac{\sin^4 x}{(\sin^4 x + \cos^4 x)}\,dx = ?$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 1 (d) 0

46. $\int_0^{\pi/2} \frac{\cos^{1/4} x}{(\sin^{1/4} x + \cos^{1/4} x)} dx = ?$

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) none of these

47. $\int_0^{\pi/2} \frac{\sin^n x}{(\sin^n x + \cos^n x)} dx = ?$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) 1 (d) 0

48. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(\sqrt{\cot x} + \sqrt{\tan x})} dx = ?$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) none of these

49. $\int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sqrt[3]{\tan x} + \sqrt[3]{\cot x})} dx = ?$

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

50. $\int_0^{\pi/2} \frac{1}{(1 + \tan x)} dx = ?$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

51. $\int_0^{\pi/2} \frac{1}{(1 + \sqrt{\cot x})} dx = ?$

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

52. $\int_0^{\pi/2} \frac{1}{(1 + \tan^3 x)} dx = ?$

(a) $\frac{\pi}{4}$ (b) 0 (c) $\frac{\pi}{2}$ (d) none of these

53. $\int_0^{\pi/2} \frac{\sec^5 x}{(\sec^5 x + \operatorname{cosec}^5 x)} dx = ?$

(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) π

54. $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{(1+\sqrt{\cot x})} dx = ?$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 0 (d) 1

55. $\int_0^{\pi/2} \frac{\tan x}{(1+\tan x)} dx = ?$

(a) 0 (b) 1 (c) $\frac{\pi}{4}$ (d) π

56. $\int_{-\pi}^{\pi} x^4 \sin x \, dx = ?$

(a) 2π (b) π (c) 0 (d) none of these

57. $\int_{-\pi}^{\pi} x^3 \cos^3 x \, dx = ?$

(a) π (b) $\frac{\pi}{4}$ (c) 2π (d) 0

58. $\int_{-\pi}^{\pi} \sin^5 x \, dx = ?$

(a) $\frac{3\pi}{4}$ (b) 2π (c) $\frac{5\pi}{16}$ (d) 0

59. $\int_{-1}^{-2} x^3(1-x^2) \, dx = ?$

(a) $-\frac{40}{3}$ (b) $\frac{40}{3}$ (c) $\frac{5}{6}$ (d) 0

60. $\int_{-a}^{a} \log\left(\frac{a-x}{a+x}\right) dx = ?$

(a) $2a$ (b) a (c) 0 (d) 1

61. $\int_{-\pi}^{\pi} (\sin^{61} x + x^{123}) \, dx = ?$

(a) 2π (b) 0 (c) $\frac{\pi}{2}$ (d) 125π

62. $\int_{-\pi}^{\pi} \tan x \, dx = ?$

(a) 2 (b) $\frac{1}{2}$ (c) -2 (d) 0

63. $\int_{-1}^{1} \log\,(x+\sqrt{x^2+1})\,dx = ?$

(a) $\log \frac{1}{2}$ (b) $\log\,2$ (c) $\frac{1}{2}\log\,2$ (d) 0

64. $\int_{-\pi/2}^{\pi/2} \cos x\,dx = ?$

(a) 0 (b) 2 (c) –1 (d) none of these

65. $\int_{0}^{a} \frac{\sqrt{x}}{(\sqrt{x}+\sqrt{a-x})}\,dx = ?$

(a) $\frac{a}{2}$ (b) $2a$ (c) $\frac{2a}{3}$ (d) $\frac{\sqrt{a}}{2}$

66. $\int_{0}^{\pi/4} \log\,(1+\tan\,x)\,dx = ?$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4}\log\,2$ (c) $\frac{\pi}{8}\log\,2$ (d) 0

67. $\int_{-a}^{a} f(x)\,dx = ?$

(a) $2\int_{0}^{a}\{f(x)+f(-x)\}\,dx$ (b) $2\int_{0}^{a}\{f(x)-f(-x)\}\,dx$

(c) $\int_{0}^{a}\{f(x)+f(-x)\}\,dx$ (d) none of these

68. Let $[x]$ denote the greatest integer less than or equal to x.
Then, $\int_{0}^{1.5}[x]\,dx = ?$

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 2 (d) 3

69. Let $[x]$ denote the greatest integer less than or equal to x.
Then, $\int_{-1}^{1}[x]\,dx = ?$

(a) –1 (b) 0 (c) $\frac{1}{2}$ (d) 2

70. $\int_{1}^{2}|x^2-3x+2|\,dx = ?$

(a) $\frac{-1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

71. $\int_{\pi}^{2\pi} |\sin x|\, dx = ?$

(a) 0 (b) 1 (c) 2 (d) none of these

72. $\int_{0}^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\, dx = ?$

(a) $\frac{1}{2}(\pi - \log 2)$ (b) $\left(\frac{\pi}{2} - 2\log 2\right)$

(c) $\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$ (d) none of these

73. $\int_{0}^{1} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = ?$

(a) $\frac{1}{2}(\pi - \log 2)$ (b) $\left(\frac{\pi}{2} - \log 2\right)$ (c) $(\pi - 2\log 2)$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (d)	**6.** (a)	**7.** (d)	8 (a)	**9.** (c)	**10.** (a)
11. (c)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (b)	**16.** (a)	**17.** (c)	**18.** (c)	**19.** (a)	**20.** (c)
21. (b)	**22.** (a)	**23.** (c)	**24.** (b)	**25.** (b)	**26.** (a)	**27.** (b)	**28.** (d)	**29.** (c)	**30.** (b)
31. (c)	**32.** (b)	**33.** (c)	**34.** (a)	**35.** (c)	**36.** (a)	**37.** (b)	**38.** (c)	**39.** (d)	**40.** (a)
41. (a)	**42.** (b)	**43.** (d)	**44.** (b)	**45.** (a)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (b)	**50.** (c)
51. (b)	**52.** (a)	**53.** (c)	**54.** (a)	**55.** (c)	**56.** (c)	**57.** (d)	**58.** (d)	**59.** (d)	**60.** (c)
61. (b)	**62.** (d)	**63.** (d)	**64.** (b)	**65.** (a)	**66.** (c)	**67.** (c)	**68.** (a)	**69.** (a)	**70.** (b)
71. (c)	**72.** (c)	**73.** (b)							

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. $I = \int_{1}^{4} x^{3/2} dx = \left[\frac{2}{5}x^{5/2}\right]_{1}^{4} = \frac{62}{5} = 12.4.$

2. $I = \int_{0}^{2} (6x+4)^{1/2} dx = \left[\frac{2}{3}\cdot\frac{(6x+4)^{3/2}}{6}\right]_{0}^{2} = \frac{56}{9}.$

3. $I = \int_{0}^{1} (5x+3)^{-1/2} dx = \left[2\cdot\frac{(5x+3)^{1/2}}{5}\right]_{0}^{1} = \frac{2}{5}(\sqrt{8} - \sqrt{3}).$

4. $I = \int_{0}^{1} \frac{dx}{(1+x^2)} = [\tan^{-1}x]_{0}^{1} = (\tan^{-1}1 - \tan^{-1}0) = \frac{\pi}{4}.$

5. $I=\int_0^2 \frac{dx}{\sqrt{4-x^2}}=\left[\sin^{-1}\frac{x}{2}\right]_0^2=(\sin^{-1}1-\sin^{-1}0)=\frac{\pi}{2}.$

6. Put $(1+x^2)=t$ and $2x\,dx=dt$.

   $[x=\sqrt{3}\ \Rightarrow\ t=4]$ and $[x=\sqrt{8}\ \Rightarrow\ t=9]$

   $\therefore\quad I=\int_4^9 \sqrt{t}\,dt=\left[\frac{1}{2}\times\frac{2}{3}t^{3/2}\right]_4^9=\frac{1}{3}(27-8)=\frac{19}{3}.$

7. Put $x^4=t$ and $4x^3dx=dt$.

   $[x=0\ \Rightarrow\ t=0]$ and $[x=1\ \Rightarrow\ t=1]$

   $\therefore\quad I=\frac{1}{4}\int_0^1\frac{dt}{(1+t^2)}=\frac{1}{4}[\tan^{-1}t]_0^1=\frac{1}{4}[\tan^{-1}1-\tan^{-1}0]=\left(\frac{1}{4}\times\frac{\pi}{4}\right)=\frac{\pi}{16}.$

8. Put $\log x=t$ and $\frac{1}{x}dx=dt$.

   $[x=1\ \Rightarrow\ t=0]$ and $[x=e\ \Rightarrow\ t=\log e=1]$

   $\therefore\quad I=\int_0^1 t^2dt=\left[\frac{t^3}{3}\right]_0^1=\frac{1}{3}.$

9. $I=\int_{\pi/4}^{\pi/2}\cot x\,dx=[\log\sin x]_{\pi/4}^{\pi/2}=\left[\log 1-\log\frac{1}{\sqrt{2}}\right]=\frac{1}{2}\log 2.$

10. $I=\int_0^{\pi/2}(\sec^2x-1)\,dx=[\tan x-x]_0^{\pi/2}=\left(1-\frac{\pi}{4}\right).$

11. $I=\frac{1}{2}\int_0^{\pi/2}2\cos^2x\,dx=\frac{1}{2}\int_0^{\pi/2}(1+\cos 2x)=\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi/2}=\frac{\pi}{4}.$

12. $I=\int_{\pi/3}^{\pi/2}\operatorname{cosec}x\,dx=\left[\log\tan\frac{x}{2}\right]_{\pi/3}^{\pi/2}=\left(\log 1-\log\frac{1}{\sqrt{3}}\right)=\frac{1}{2}\log 3.$

13. $I=\int_0^{\pi/2}\cos^2x\cdot\cos x\,dx=\int_0^{\pi/2}(1-\sin^2x)\cos x\,dx=\int_0^1(1-t^2)\,dt$, where $\sin x=t$.

14. $I=\int_0^{\pi/4}e^{\tan x}\sec^2x\,dx=[e^t dt]_0^1$, where $\tan x=t$.

15. Put $\sin x=t$ and $\cos x\,dx=dt$.

    $[x=0\ \Rightarrow\ t=0]$ and $\left[x=\frac{\pi}{2}\ \Rightarrow\ t=1\right]$.

    $\therefore\quad I=\int_0^1\frac{dt}{(1+t^2)}=[\tan^{-1}t]_0^1=\frac{\pi}{4}.$

16. Put $\frac{1}{x}=t$ and $\frac{1}{x^2}dx=-dt$.

    $\left[x=\frac{1}{\pi}\ \Rightarrow\ t=\pi\right]$ and $\left[x=\frac{2}{\pi}\ \Rightarrow\ t=\frac{\pi}{2}\right]$

$$\therefore \quad I = -\int_{\pi}^{\pi/2} \sin t\, dt = \int_{\pi/2}^{\pi} \sin t\, dt = [-\cos t]_{\pi/2}^{\pi} = 1.$$

17. $$I = \int_0^{\pi} \frac{(1-\sin x)}{(1-\sin^2 x)}\, dx = \int_0^{\pi} \frac{(1-\sin x)}{\cos^2 x}\, dx$$

$$= \int_0^{\pi} [\sec^2 x - \sec x \tan x]\, dx = [\tan x - \sec x]_0^{\pi} = 2.$$

18. $$I = \int_0^{\pi/2} (\sin x)^{3/2} \cos^2 x \cos x\, dx = \int_0^1 t^{3/2}(1-t^2)\, dt, \text{ where } \sin x = t$$

$$= \int_0^1 (t^{3/2} - t^{7/2})\, dt = \left[\frac{2}{5}t^{5/2} - \frac{2}{9}t^{9/2}\right]_0^1 = \left(\frac{2}{5} - \frac{2}{9}\right) = \frac{8}{45}.$$

19. $$I = \int_0^1 e^x \left\{\frac{(1+x)-1}{(1+x)^2}\right\} dx = \int_0^1 e^x \left\{\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right\} dx$$

$$= \int_0^1 e^x \{f(x) + f'(x)\}\, dx, \quad \text{where } f(x) = \frac{1}{(1+x)}$$

$$= [e^x f(x)]_0^1 = \left[e^x \cdot \frac{1}{(1+x)}\right]_0^1 = \left(\frac{e}{2} - 1\right).$$

20. $$I = \int_0^{\pi/2} e^x \left\{\frac{1}{(1+\cos x)} + \frac{\sin x}{(1+\cos x)}\right\} dx$$

$$= \int_0^{\pi/2} e^x \left\{\frac{1}{2\cos^2(x/2)} + \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right\} dx$$

$$= \int_0^{\pi/2} e^x \left\{\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right\} dx = \int_0^{\pi/2} e^x \{f(x) + f'(x)\}\, dx$$

$$= \left[e^x \tan\frac{x}{2}\right]_0^{\pi/2} = e^{\pi/2}.$$

21. $$I = \int_0^{\pi/4} \sqrt{\cos^2 x + \sin^2 x + 2\sin x \cos x}\, dx = \int_0^{\pi/4} (\cos x + \sin x)\, dx$$

$$= [\sin x - \cos x]_0^{\pi/4} = 1.$$

22. $$I = \int_0^{\pi/2} \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int_0^{\pi/2} \cos x\, dx = \sqrt{2}[\sin x]_0^{\pi/2} = \sqrt{2}.$$

23. On dividing $(-x+1)$ by $(x+1)$ we get:

$$I = \int_0^1 \left\{-1 + \frac{2}{(x+1)}\right\} dx = [-x + 2\log(x+1)]_0^1 = (2\log 2 - 1).$$

24. $I = \frac{1}{2}\cdot\int_0^{\pi/2}(1-\cos 2x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2} = \frac{\pi}{4}\cdot$

25. $I = \frac{1}{2}\int_0^{\pi/6}[\cos 3x + \cos x]\,dx = \frac{1}{2}\left[\frac{\sin 3x}{3} + \sin x\right]_0^{\pi/6} = \frac{15}{12}\cdot$

26. $I = \frac{1}{2}\int_0^{\pi/2}[\cos(2x-x) - \cos(2x+x)]\,dx = \frac{1}{2}\int_0^{\pi/2}(\cos x - \cos 3x)\,dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right]_0^{\pi/2} = \frac{1}{2}\left(1+\frac{1}{3}\right) = \frac{2}{3}\cdot$

27. $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$\therefore \quad I = \frac{1}{2}\int_0^{\pi}(\sin 5x - \sin x)\,dx = \frac{1}{2}\cdot\left[\frac{-\cos 5x}{5} + \cos x\right]_0^{\pi} = \frac{-4}{5}\cdot$

28. $I = \int_0^1 \frac{dx}{\left(e^x + \frac{1}{e^x}\right)} = \int_0^1 \frac{e^x dx}{(e^{2x}+1)} = \int_1^e \frac{dt}{(t^2+1)}$, where $e^x = t$.

$= [\tan^{-1} t]_1^e = \left(\tan^{-1} e - \frac{\pi}{4}\right)\cdot$

29. Put $x = t^2$ and $dx = 2t\,dt$.

$[x = 0 \Rightarrow t = 0]$ and $[x = 9 \Rightarrow t^2 = 9 \Rightarrow t = 3]$

$\therefore \quad I = \int_0^3 \frac{2t}{(1+t)}dt = \int_0^3 \left\{2 - \frac{2}{(1+t)}\right\}dt = [2t - 2\log(1+t)]_0^3 = (6 - 2\log 4).$

30. Integrating by parts, we get:

$$I = \int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2}(1\cdot\sin x)\,dx = \frac{\pi}{2} + [\cos x]_0^{\pi/2} - \left(\frac{\pi}{2} - 1\right)\cdot$$

31. $I = \int_0^1 \frac{dx}{\left\{\left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}\right\}} = \int_0^1 \frac{dx}{\left\{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right\}} = \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\right]_0^1 = \frac{\pi}{3\sqrt{3}}\cdot$

32. $I = \int_0^1 \left\{\frac{\sqrt{1-x}}{\sqrt{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}}\right\}dx = \int_0^1 \frac{(1-x)}{\sqrt{1-x^2}}dx.$

$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}dx = [\sin^{-1} x]_0^1 + \frac{1}{2}\int_1^0 \frac{1}{\sqrt{t}}dt$, where $(1-x^2) = t$

$= [\sin^{-1} 1 - \sin^{-1} 0] - \frac{1}{2}\cdot\int_0^1 \frac{1}{\sqrt{t}}dt = \left(\frac{\pi}{2} - 0\right) - \frac{1}{2}\cdot\left[\frac{t^{1/2}}{(1/2)}\right]_0^1 = \left(\frac{\pi}{2} - 1\right)\cdot$

33. $I = \int_0^1 \left(-1 + \frac{2}{x+1}\right)dx = [-x + 2\log|x+1|]_0^1 = (2\log 2 - 1).$

34. Put $x = a\cos\theta$ and $dx = -a\sin\theta\, d\theta$.

$[x = -a \Rightarrow \theta = \pi]$ and $[x = a \Rightarrow \theta = 0]$.

$$\therefore \quad I = \int_{\pi}^{0} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot (-a\sin\theta)\, d\theta = a\int_{0}^{\pi} \left\{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}\right\}^{1/2} 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta$$

$$= a\int_{0}^{\pi} 2\sin^2\frac{\theta}{2}\, d\theta = a\int_{0}^{\pi}(1-\cos\theta)\, d\theta = a\pi.$$

35. Put $x = \sqrt{2}\sin t$ and $dx = \sqrt{2}\cos t\, dt$.

$[x = 0 \Rightarrow t = 0]$ and $\left[x = \sqrt{2} \Rightarrow t = \frac{\pi}{2}\right]$

$$\therefore \quad I = \int_{0}^{\pi/2} \sqrt{2 - 2\sin^2 t} \cdot \sqrt{2}\cos t\, dt = 2\int_{0}^{\pi/2} \cos^2 t\, dt = 2\int_{0}^{\pi/2} \frac{1}{2}(1+\cos 2t)\, dt$$

$$= \left[t + \frac{\sin 2t}{2}\right]_0^{\pi/2} = \frac{\pi}{2}.$$

36. $$I = \int_{-2}^{0} |x|\, dx + \int_{0}^{2} |x|\, dx = \int_{-2}^{0} -x\, dx + \int_{0}^{2} x\, dx.$$

$$= \left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^2 = 0 - (-2) + (2-0) = 4.$$

37. Put $2x - 1 = t$ and $dx = \frac{1}{2}dt$.

$[x = 0 \Rightarrow t = -1]$ and $[x = 1 \Rightarrow t = 1]$

$$\therefore \quad I = \frac{1}{2}\int_{-1}^{1} |t|\, dt = \frac{1}{2}\left\{\int_{-1}^{0} -t\, dt + \int_{0}^{1} t\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-t^2}{2}\right]_{-1}^{0} + \left[\frac{t^2}{2}\right]_0^1\right\} = \frac{1}{2}\left\{\left(0 + \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right)\right\} = \frac{1}{2}.$$

38. Put $2x + 1 = t$ and $dx = \frac{1}{2}dt$

$[x = -2 \Rightarrow t = -3]$ and $[x = 1 \Rightarrow t = 3]$

$$\therefore \quad I = \frac{1}{2}\int_{-3}^{3} |t|\, dt = \frac{1}{2}\left\{\int_{-3}^{0} (-t)\, dt + \int_{0}^{3} t\, dt\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-t^2}{2}\right]_{-3}^{0} + \left[\frac{t^2}{2}\right]_0^3\right\} = \frac{1}{2}\left[\left\{0 - \left(\frac{-9}{2}\right)\right\} + \left(\frac{9}{2} - 0\right)\right] = \frac{9}{2}.$$

39. $$I = \int_{-2}^{0} \frac{-x}{x}\, dx + \int_{0}^{1} \frac{x}{x}\, dx = \int_{-2}^{0} -dx + \int_{0}^{1} dx = [-x]_{-2}^{0} + [x]_0^1 = (-2+1) = -1.$$

40. $$I = \int_{-a}^{0} x(-x)\, dx + \int_{0}^{a} x\cdot x\, dx = \int_{-a}^{0} -x^2 dx + \int_{0}^{a} x^2 dx$$

$$= \left[\frac{-x^3}{3}\right]_{-a}^{0} + \left[\frac{x^3}{3}\right]_0^a = +\frac{1}{3}(-a)^3 + \frac{a^3}{3} = 0.$$

41. $I = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx.$

$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 - (0 - 1) = 2.$

42. $I = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} -\sin x\,dx.$

$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} = (2 + 2) = 4.$

43. $I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)}\,dx$ ... (i)

$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left\{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right\}}\,dx = \int_0^{\pi/2} \frac{\cos x}{(\sin x + \cos x)}\,dx$ ... (ii)

$\therefore \quad 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}.$

Q. 44 to 49: Similarly, each of the questions from 44 to 49 may be attempted.

50. $I = \int_0^{\pi/2} \frac{\cos x}{(\cos x + \sin x)}\,dx.$

51. $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx.$

52. $I = \int_0^{\pi/2} \frac{\cos^3 x}{(\cos^3 x + \sin^3 x)}\,dx.$

54. $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{(\sqrt{\sin x} + \sqrt{\cos x})}\,dx.$

55. $I = \int_0^{\pi/2} \frac{\sin x}{(\cos x + \sin x)}\,dx.$

56. $f(x) = x^4 \sin x.$

$\therefore \quad f(-x) = (-x)^4 \sin(-x) = -(x^4 \sin x) = -f(x).$

$\therefore \quad f(x)$ is odd and hence $I = 0.$

57. $f(x) = x^3 \cos^3 x.$

$\therefore \quad f(-x) = (-x)^3[\cos(-x)]^3 = -x^3(\cos x)^3 = -(x^3 \cos^3 x) = -f(x)$

$\therefore \quad f(x)$ is odd and hence $I = 0.$

58. $f(x) = \sin^5 x.$

$f(-x) = [\sin(-x)]^5 = (-\sin x)^5 = -(\sin^5 x) = -f(x).$

$\therefore \quad f(x)$ is odd and hence $I = 0.$

59. $f(x) = x^3(1 - x^2) = (x^3 - x^5).$

$f(-x) = (-x)^3 - (-x)^5 = -x^3 - (-x^5) = -x^3 + x^5 = -x^3(1 - x^2) = -f(x).$

$\therefore \quad f(x)$ is odd and hence $I = 0.$

60. $f(x) = \log\left(\frac{a-x}{a+x}\right)$.

$\Rightarrow \quad f(-x) = \log\left(\frac{a+x}{a-x}\right) = \log\left\{\left(\frac{a-x}{a+x}\right)^{-1}\right\} = -\log\left(\frac{a-x}{a+x}\right) = -f(x)$

$\therefore \quad f(x)$ is odd and hence $I = 0$.

61. $f(x) = \sin^{61} x + x^{123}$

$\Rightarrow \quad f(-x) = [\sin(-x)]^{61} + (-x)^{123} = (-\sin x)^{61} - x^{123} = -\sin^{61} x - x^{123}$

$= -[\sin^{61} x + x^{123}] = -f(x)$.

$\therefore \quad f(x)$ is odd and hence $I = 0$.

62. $f(x) = \tan x \Rightarrow f(-x) = \tan(-x) = -\tan x = -f(x)$.

$\therefore \quad f(x)$ is odd and hence $I = 0$.

63. Let $f(x) = \log(x + \sqrt{x^2+1})$. Then,

$f(-x) = \log(-x + \sqrt{x^2+1})$

$= \log\left\{\frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{(\sqrt{x^2+1} + x)}\right\} = \log\left\{\frac{1}{(\sqrt{x^2+1} + x)}\right\}$

$= \log(x + \sqrt{x^2+1})^{-1} = -\log(x + \sqrt{x^2+1}) = -f(x)$

$\therefore \quad f(x)$ is odd and hence $I = 0$.

64. $f(x) = \cos x \Rightarrow f(-x) = \cos(-x) = \cos x = f(x)$.

$\therefore \quad f(x)$ is an even function.

$\therefore \quad I = 2\int_0^{\pi/2} \cos x\, dx = 2[\sin x]_0^{\pi/2} = (2 \times 1) = 2.$

65. $I = \int_0^a \frac{\sqrt{x}}{(\sqrt{x} + \sqrt{a-x})} dx$ and $I = \int_0^a \frac{\sqrt{a-x}}{\{\sqrt{a-x} + \sqrt{a-(a-x)}\}} dx$

$\therefore \quad 2I = \int_0^a \frac{(\sqrt{x} + \sqrt{a-x})}{(\sqrt{x} + \sqrt{a-x})} dx = \int_0^a dx = [x]_0^a = a \Rightarrow I = \frac{a}{2}$.

66. $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$ ... (i)

$I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx = \int_0^{\pi/4} \log\left\{1 + \frac{1 - \tan x}{1 + \tan x}\right\} dx$

$= \int_0^{\pi/4} \{\log 2 - \log(1 + \tan x)\} dx = \int_0^{\pi/4} (\log 2)\, dx - I$

$\Rightarrow \quad 2I = (\log 2)[x]_0^{\pi/4} = \frac{\pi}{4}(\log 2) \Rightarrow I = \frac{\pi}{8}(\log 2).$

67. We have: $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x) + f(-x)\}\,dx.$

68. $\{0 \le x < 1 \Rightarrow [x] = 0\}$ and $\{1 \le x < 1.5\} \Rightarrow [x] = 1$

$$\therefore \quad I = \int_0^1 [x]\,dx + \int_1^{1.5} [x]\,dx = \int_0^1 0\,dx + \int_0^{1.5} 1 \cdot dx$$

$$= 0 + [x]_1^{1.5} = (1.5 - 1) = 0.5 = \frac{1}{2}.$$

69. $\{-1 \le x < 0 \Rightarrow [x] = -1\}$ and $\{0 \le x < 1 \Rightarrow [x] = 0\}$

$$\therefore \quad I = \int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx = \int_{-1}^{0} (-1)\,dx + \int_0^1 0 \cdot dx$$

$$= [-x]_{-1}^{0} = 0 - \{-(-1)\} = -1.$$

70. $(x^2 - 3x + 2) = (x - 1)(x - 2)$

$1 \le x < 2 \Rightarrow (x^2 - 3x + 2) \le 0$

$$\Rightarrow |x^2 - 3x + 2| = -(x^2 - 3x + 2)$$

$$\therefore \quad I = \int_1^2 -(x^2 - 3x + 1)\,dx = \int_1^2 (-x^2 + 3x - 2)\,dx$$

$$= \left[\frac{-x^3}{3} + \frac{3x^2}{2} - 2x\right]_1^2 = \left(\frac{-2}{3} + \frac{5}{6}\right) = \frac{1}{6}.$$

71. $\pi \le x < 2\pi \Rightarrow \sin x \le 0 \Rightarrow |\sin x| = -\sin x$

$$\therefore \quad I = \int_{\pi}^{2\pi} -\sin x\,dx = [\cos x]_{\pi}^{2\pi} = 2.$$

72. Put $x = \sin t$ and $dx = \cos t\,dt$

$[x = 0 \Rightarrow \sin t = 0 \Rightarrow t = 0]$ and $\left[x = \frac{1}{\sqrt{2}} \Rightarrow \sin t = \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\pi}{4}\right]$

$$\therefore \quad I = \int_0^{\pi/4} \frac{t \cos t}{\cos^3 t}\,dt = \int_0^{\pi/4} t \sec^2 t = [t \tan t]_0^{\pi/4} - \int_0^{\pi/4} \tan t\,dt$$

$$= \frac{\pi}{4} + [\log \cos t]_0^{\pi/4} = \frac{\pi}{4} + \log \frac{1}{\sqrt{2}} = \left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$$

73. Put $x = \tan t$ and $dx = \sec^2 t\,dt$.

$[x = 0 \Rightarrow t = 0]$ and $\left[x = 1 \Rightarrow \tan t = 1 \Rightarrow t = \frac{\pi}{4}\right]$

$$\therefore \quad I = \int_0^{\pi/4} \sin^{-1}(\sin 2t) \sec^2 t\,dt = 2\int_0^{\pi/4} t \sec^2 t\,dt$$

$$= 2\left[[t \tan t]_0^{\pi/4} - \int_0^{\pi/4} \tan t\,dt\right] = 2\left\{\frac{\pi}{4} + [\log \cos t]_0^{\pi/4}\right\}$$

$$= \frac{\pi}{2} + 2\log \frac{1}{\sqrt{2}} = \left(\frac{\pi}{2} - \log 2\right).$$